# How Likely is Contagion in Financial Networks?

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# How Likely is Contagion in Financial Networks?

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## Abstract

Interconnections among financial institutions create potential channels for contagion and amplification of shocks to the financial system. We estimate the extent to which interconnections increase expected losses and defaults under a wide range of shock distributions. In contrast to most work on financial networks, we assume only minimal information about network structure and rely instead on information about the individual institutions that are the nodes of the network. The key node-level quantities are asset size, leverage, and a financial connectivity measure given by the fraction of a financial institution's liabilities held by other financial institutions. We combine these measures to derive explicit bounds on the potential magnitude of network effects on contagion and loss amplification. Spillover effects are most significant when node sizes are heterogeneous and the originating node is highly leveraged and has high financial connectivity. Our results also highlight the importance of mechanisms that go beyond simple spillover effects to magnify shocks; these include bankruptcy costs, and mark-to-market losses resulting from credit quality deterioration or a loss of confidence. We illustrate the results with data on the European banking system.

Keywords: *systemic risk, contagion, financial network*

JEL: D85, G21

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# 1    Introduction

The interconnectedness of the modern financial system is widely viewed as having been a key contributing factor to the recent financial crisis. Due to the complex web of links between institutions, stresses to one part of the system can spread to others, leading to a system-wide threat to financial stability. Specific instances include the knock-on effects of the Lehman bankruptcy, potential losses to counterparties that would have resulted from a failure of the insurance company AIG, and more recently the exposure of European banks to the risk of sovereign default by some European countries. These and other examples highlight concerns that interconnectedness could pose a significant threat to the stability of the financial system.[1] Moreover there is a growing body of research that shows how this can happen in a theoretical sense.[2]

Although it is intuitively clear that interconnectedness has some effect on the transmission of shocks, it is less clear whether it significantly increases the likelihood and magnitude of losses compared to a financial system that is not interconnected. The contribution of this paper is to provide a general framework for analyzing this question. In contrast to much of the prior literature we do not subject the network to ad hoc shocks of different sizes. Instead we assume a full-fledged shock distribution and analyse the probability of default cascades and consequent losses of value that are attributable to network connections. A second distinguishing feature of our analysis is that, instead of estimating the absolute magnitude of default probabilities and losses, we estimate how much larger these quantities are in a networked system as compared to a similar system in which all links between financial institutions have been severed. In other words we estimate the extent to which defaults and losses are magnified by the interbank network over and above the original shocks.

It turns out that one can derive general bounds on the impact of the network with very little information about the network topology: our bounds hold independently of the degree distribution, node centrality, average path length, and so forth. This topology-free property of our results is one of the main contributions of the paper. We also show that these bounds hold for a wide range of distributions, including beta, exponential, normal, and many others. This robustness is important because detailed information about interbank liabilities is often unavailable and the exact form of the shock distributions is subject to considerable uncertainty. We are not claiming that the network topology is inconsequential, but that one can derive useful bounds on the financial system's susceptibility to contagion without knowing the details of the topology.

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[1] See, for example, Bank of England 2011, International Monetary Fund 2012, and Office of Financial Research 2012.

[2] See in particular Allen and Gale (2000), Upper and Worms (2002), Degryse and Nguyen (2004), Goodhart, Sunirand, and Tsomocos (2004), Elsinger, Lehar, and Summer (2006), Allen and Babus (2009), Gai and Kapadia (2010), Gai, Haldane, and Kapadia (2011), Haldane and May (2011), Upper (2011), Georg (2013), Rogers and Veraart (2012), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and Elliott, Golub, and Jackson (2014).

The starting point for our analysis is the elegant framework of Eisenberg and Noe (2001). Their model specifies a set of nodes that represent financial institutions together with the obligations between them. Given an initial shock to the balance sheets of one or more nodes, one can compute a set of payments that clear the network; that is, it provides a consistent way of valuing all the nodes conditional on an arbitrary shock to the system. This framework is very useful for analyzing how losses propagate through the financial system. A concrete example would be delinquencies in mortgage payments: if some fraction of a bank's mortgages are delinquent and it has insufficient reserves to cover the shortfall, then it will be unable to pay its creditors in full, who may be unable to pay their creditors in full, and so forth. The original shortfall in payments can cascade through the system, causing more and more banks to default through a domino effect. The Eisenberg-Noe framework shows how to compute a set of payments that clear the network, and it identifies which nodes default as a result of an initial shock to the system. The number and magnitude of such defaults depend on thenetwork topolgoy, and there is now a substantial literature characterizing those structures that tend to propagate default or alternatively that tend to dampen it (Gai and Kapadia 2010,Gai, Haldane, and Kapadia 2011, Haldane and May, 2011, Acemoglu, Ozdaglar, and Tahbaz-Salehi 2015, and Elliott, Golub, and Jackson 2014).

One limitation of the Eisenberg-Noe framework, as with most models of financial networks, is that it does not provide an account of link formation — that is, it does not model the dynamic process by which financial institutions enter into obligations to one another in the first place. This underscores the importance of having an estimation framework that does not rely too heavily on the specific features of the network, which is the subject of the present paper. We take the balance sheets of individual financial institutions as given and estimate how much they contribute to systemic effects over and above the impact of the initial shocks to asset values. In particular we shall examine the following two questions: How likely is it that a given set of nodes will default due to contagion from another node, as compared to the likelihood that they default from direct shocks to their own assets from sources outside the financial system, such as households and nonfinancial firms? And how much does the network increase the probability and magnitude of losses compared to a situation where there are no connections?

To compare systems with and without interconnections, we proceed as follows. First, we define our nodes to be financial institutions that borrow and lend on a significant scale, which together with their obligations to one another constitute the *financial network*. In addition, such institutions borrow and lend to the nonfinancial sector, which is composed of investors, households, and nonfinancial firms. We compare this system to one without connections that is constructed as follows. We remove all of the obligations between the financial nodes while keeping their links with the nonfinancial sector unchanged. We also keep node equity values as before by creating, for each node, a fictitious outside asset (or liability) whose value equals the net value of the connections at that node that were removed. We then apply

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the same shock distributions to both systems, with the shocks to real assets originating in the external sector and the fictitious assets (if any) assumed to be impervious to shocks. We can ascertain how much the network connections contribute to increased defaults and losses by comparing the outcomes in the two systems.

One might suppose that such a comparison is sensitive to the choice of shock distribution, but this turns out not to be the case: we show how to compute general bounds on the increased losses attributable to network connections that hold under a wide variety of distributions, including the beta, exponential, normal and many others. The bounds also hold whether the shocks are independent or positively associated and thus capture the possibility that institutions have portfolios that are exposed to common shocks (see for example Caccioli et al. 2012).

Two key findings emerge from this analysis, one concerning the probability of default cascades and the other concerning the expected losses from such cascades. We begin by computing the probability that default at a given node causes defaults at other nodes (via network spillovers), and compare this with the probability that all of these nodes default by direct shocks to their outside assets with no network transmission. We then derive a general formula that shows when the latter probability is larger than the former, in which case we say that contagion is *weak*. This characterization shows explicitly that substantial heterogeneity in node sizes makes a network more vulnerable to contagion through pure spillover effects. The network is particularly vulnerable to contagion when the originating node is large, highly leveraged, and, crucially, has a relatively high proportion of its obligations held by other financial institutions as opposed to the nonfinancial sector, what we will call high *financial connectivity*. These three factors — size, leverage, and financial connectivity — determine a *contagion index* for each institution that measures the potential impact of its failure on the rest of the financial sector.

Second, we apply our framework to estimate the expected system-wide loss in asset values that results from shocks that originate outside the financial sector. We derive a simple formula that compares the the additional expected losses generated by the network with the expected losses that occur when the network links are severed. It turns out that the losses attributable to the network are typically quite modest under a wide range of shock distributions. Here, again, the network effect is highly dependent on the level of financial connectivity.

We emphasize that these results do not imply that *all* forms of network contagion are unimportant; rather they demonstrate that simple spillover or "domino" effects have only limited impact. These findings are consistent with the empirical and simulation literature on network stress testing, which finds that contagion is quite difficult to generate through the interbank spillover of losses (Degryse and Nguyen 2004, Elsinger, Lehar, and Summer 2006, Furfine 2003, Georg 2013, Nier et al. 2007). Put differently, our results show that contagion through spillover effects becomes most significant under the

conditions described in Yellen (2013), when financial institutions inflate their balance sheets by increasing leverage and expanding interbank claims backed by a fixed set of real assets.

Indeed our results suggest that additional channels, aside from pure spillover effects, are needed to generate substantial losses from contagion. One such channel involves fire sales, in which firms dump assets on the market in order to cover their losses.[3] Another channel is the drying up of liquidity, which results when the default of one institution heightens uncertainty about the health of others, leading to a general tightening of credit. More generally, financial institutions may respond to changing market conditions in a variety of ways that exacerbate the impact of an initial negative shock and result in contagion. Although some of these dynamic effects are difficult to capture within an essentially static network model, there are two important sources of contagion that can be incorporated into our framework, namely, bankruptcy costs and losses of confidence.

Bankruptcy costs magnify the costs associated with default both directly, through costs like legal fees, and indirectly through delays in payments to creditors and disruptions to the provision of financial intermediation services necessary to the real economy. We model these effects in reduced form through a multiplier on losses when a node defaults.[4] This approach allows us to estimate how much the probability of contagion, and the expected losses induced by contagion, increase as a function of bankruptcy costs. A somewhat surprising finding is that bankruptcy costs must be quite large in order to have an appreciable impact on expected losses as they propagate through the network.

A second mechanism that we believe to be of great importance is crises of confidence in the credit quality of particular firms. If a firm's *perceived* ability to pay declines for whatever reason, then so does the market value of its liabilities. In a mark-to-market regime this reduction in value can spread to other firms that hold these liabilities among their assets. In other words, the mere possibility (rather than the actuality) of a default can lead to a general and widespread decline in valuations, which may in turn trigger actual defaults through mark-to-market losses.[5] This is an important phenomenon in practice: indeed it has been estimated that mark-to-market losses from credit quality deterioration exceeded losses from outright defaults in 2007-2009.[6]

We capture this idea by re-interpreting the Eisenberg-Noe framework as a *valuation model* rather than as a clearing model. Declines in confidence about the ability to pay at some nodes can spread to other nodes through a downward revaluation of their assets. This mechanism shows how a localized

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[3]See Shleifer and Vishny (2011) for a survey and Cifuentes, Ferrucci, and Shin (2005) for an extension of the Eisenberg-Noe framework with fire sales.

[4]For other approaches see Cont, Moussa, and Santos (2010), Rogers and Veraart (2012), and Section 5.

[5]This mechanism differs from a bank run, which could also be triggered by a loss of confidence. Mark-to-market losses spread when a lender continues to extend credit, whereas a run requires withdrawal of credit. In the seminal framework of Diamond and Dybvig (1983), a run is triggered by a demand for liquidity rather than a concern about credit quality.

[6]According to the Basel Committee on Banking Supervision, for example, roughly two thirds of losses attributed to counterparty credit risk were due to mark-to-market losses and only about one third of losses were due to actual defaults. See http://www.bis.org/press/p110601.htm.

crisis of confidence can lead to widespread losses of value. Our analysis suggests that this channel of contagion is likely to be considerably more important than simple domino or spillover effects.

We briefly mention some other frameworks that have been developed to study financial contagion, in addition to those already cited. One strand of research uses co-movements in market prices, as in Acharya et al. (2017) and Adrian and Brunnermeier (2016). Another approach is based on contagion through common asset holdings and includes the work of Allen, Babus, and Carletti (2012) and Caccioli et al. (2012). There is also a large literature on *simulation* models of contagion in interbank networks; for a recent survey, see Summer (2013). Our approach differs from earlier work by providing explicit expressions for the impact of the network effects on contagion and amplification with minimal assumptions about the network structure and the distribution of shocks.

The rest of the paper is organized as follows. In Section 2 we present the basic Eisenberg-Noe framework and illustrate its operation through a series of simple examples. In Section 3 we introduce shock distributions explicitly. We then compare the probability that a given set of nodes default from simultaneous direct shocks to their outside assets, with the probability that they default indirectly by contagion from some other node. In Section 4 we examine the expected loss in value that is attributable to network contagion using the comparative framework described above. We show that one can obtain useful bounds on the losses attributable to the network with almost no knowledge of the specific network topology and under very general assumptions about the shock distributions. In Section 5 we introduce bankruptcy costs and show how to extend the preceding analysis to this case. Section 6 examines the effects of a deterioration in confidence in one or more institutions, such as occurred in the 2008-09 financial crisis. We show how such a loss of confidence can spread through the entire system due to mark-to-market declines in asset values. In the Appendix we illustrate the application of these ideas to the European banking system using data from European Banking Authority (2011).

## 2 Measuring systemic risk

### 2.1 The Eisenberg-Noe framework

The network model proposed by Eisenberg and Noe (2001) has three basic ingredients: a set of $n$ nodes $N = \{1, 2, ..., n\}$, an $n \times n$ liabilities matrix $\bar{P} = (\bar{p}_{ij})$ where $\bar{p}_{ij} \geq 0$ represents the *payment due* from node $i$ to node $j$, $\bar{p}_{ii} = 0$, and a vector $c = (c_1, c_2, ..., c_n) \in R^n_+$ where $c_i \geq 0$ represents the value of *outside assets* held by node $i$ in addition to its claims on other nodes in the network. Typically $c_i$ consists of cash, securities, mortgages and other claims on entities outside the network. In addition each node $i$ may have liabilities to entities outside the network; we let $b_i \geq 0$ denote the sum of all such liabilities of $i$, which we assume have equal priority with $i$'s liabilities to other nodes in the network.

The *asset* side of node $i$'s balance sheet is given by $c_i + \sum_{j \neq i} \bar{p}_{ji}$, and the *liability* side is given by
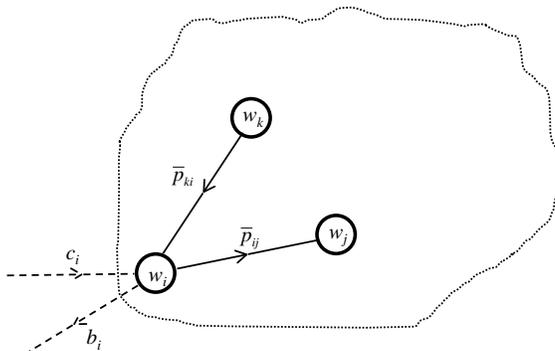
Figure 1: Node $i$ has an obligation $\bar{p}_{ij}$ to node $j$, a claim $\bar{p}_{kj}$ on node $k$, outside assets $c_i$, and outside liabilities $b_i$, for a net worth of $w_i$.

$\bar{p}_i = b_i + \sum_{j \neq i} \bar{p}_{ij}$. Its *net worth* is the difference

$$w_i = c_i + \sum_{j \neq i} \bar{p}_{ji} - \bar{p}_i. \tag{1}$$

The notation associated with a generic node $i$ is illustrated in Figure 1. Inside the network (indicated by the dotted line), node $i$ has an obligation $\bar{p}_{ij}$ to node $j$ and a claim $\bar{p}_{ki}$ on node $k$. The figure also shows node $i$'s outside assets $c_i$ and outside liabilities $b_i$. The difference between total assets and total liabilities is the node's net worth $w_i$.

Observe that $i$'s net worth is unrestricted in sign; if it is nonnegative then it corresponds to the book value of $i$'s equity. We call this "book value" because it is based on the nominal or face value of the liabilities $\bar{p}_{ji}$, rather than on "market" values that reflect the nodes' ability to pay. These market values depend on other nodes' ability to pay conditional on the realized value of their outside assets. To be specific, let each node's outside assets be subjected to a random shock that reduces the value of its outside assets, and hence its net worth. These are shocks to "fundamentals" that propagate through the network of financial obligations. Let $X_i \in [0, c_i]$ be a random shock that reduces the value of $i$'s outside assets from $c_i$ to $c_i - X_i$. After the shock, $i$'s net worth has become $w_i - X_i$. Let $F(x_1, x_2, ..., x_n)$ be the joint cumulative distribution function of these shocks; we shall consider specific classes of shock distributions in the next section. (We use $X_i$ to denote a random variable and $x_i$ to denote a particular realization.)

To illustrate the effect of a shock, we consider the numerical example in Figure 2(a), which follows the notational conventions of Figure 1. In particular, the central node has a net worth of 10 because it has 150 in outside assets, 100 in outside liabilities, and 40 in liabilities to other nodes inside the network. A shock of magnitude 10 to the outside assets erases the central node's net worth, but leaves it with just enough assets (140) to fully cover its liabilities. A shock of magnitude 80 leaves the central node with

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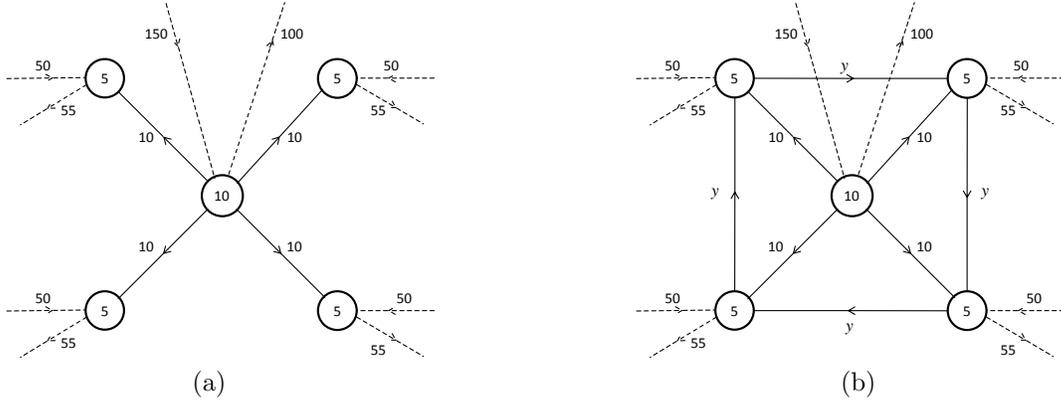
Figure 2: Two network examples.

assets of 70, half the value of its liabilities. Under a *pro rata* allocation, each liabilitity is cut in half, so each peripheral node receives a payment of 5, which is just enough to balance each peripheral node's assets and liabilities. Thus, in this case, the central node defaults but the peripheral nodes do not. A shock to the central node's outside assets greater than 80 would reduce the value of every node's assets below the value of its liabilities.

Figure 2(b) provides a more complex version of this example in which a cycle of obligations of size $y$ runs through the peripheral nodes. To handle such cases, we need the notion of a clearing vector introduced by Eisenberg and Noe (2001).

The *relative liabilities matrix* $A = (a_{ij})$ is the $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} \bar{p}_{ij}/\bar{p}_i, & \text{if } \bar{p}_i > 0 \\ 0, & \text{if } \bar{p}_i = 0 \end{cases} \tag{2}$$

Thus $a_{ij}$ is the proportion of $i$'s obligations owed to node $j$. Since $i$ may also owe entities in the external sector, $\sum_{j \neq i} a_{ij} \leq 1$ for each $i$, that is, $A$ is row substochastic.[7]

Given a shock realization $x = (x_1, x_2, ..., x_n) \geq 0$, a *clearing vector* $p(x) \in R_+^n$ is a solution to the system

$$p_i(x) = \bar{p}_i \wedge \left( \sum_j p_j(x) a_{ji} + c_i - x_i \right)_+. \tag{3}$$

As we shall subsequently show, the clearing vector is unique if the following condition holds: from every node $i$ there exists a chain of positive obligations to some node $k$ that has positive obligations to the external sector. (This amounts to saying that $A$ has spectral radius less than 1.) We shall assume that this condition holds throughout the remainder of the paper.

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[7]The row sums are all equal to 1 in Eisenberg and Noe (2001) because $b_i = 0$ in their formulation.

## 2.2 Mark-to-market values

The usual way of interpreting $p(x)$ is that it corresponds to the payments that balance the realized assets and liabilities at each node given that: i) debts take precedence over equity; and ii) all debts at a given node are written down *pro rata* when the net assets at that node (given the payments from others), is insufficient to meet its obligations. In the latter case the node is *in default*, and the *default set* is

$$D(p(x)) = \{i : p_i(x) < \bar{p}_i\}. \tag{4}$$

However, a second (and, in our setting, preferable) way of interpreting $p(x)$ is to see (3) as a *mark-to-market* valuation of all assets following a shock to the system. The nominal value $\bar{p}_i$ of node $i$'s liabilities is marked down to $p_i(x)$ as a consequence of the shock $x$, including its impact on other nodes. As in our discussion above of Figure 2(a), after marking-to-market, node $i$'s net worth is reduced from (1) to

$$c_i - x_i + \sum_{j \neq i} p_{ji}(x) - p_i(x). \tag{5}$$

The reduction in net worth reflects both the *direct* effect of the shock component $x_i$ and the *indirect* effects of the full shock vector $x$. Note, however, that this is a statement about *values*; it does not require that the payments actually be made at the end of the period. *Under this interpretation $p(x)$ provides a consistent re-valuation of the assets and liabilities of all the nodes when a shock $x$ occurs.*

As shown by Eisenberg and Noe, a solution to (3) can be constructed iteratively as follows. Given a realized shock vector $x$ (and recalling that $x_i \leq c_i$) define the mapping $\Phi : R_+^n \to R_+^n$ as follows:

$$\forall i, \quad \Phi_i(p) = \bar{p}_i \wedge \left( \sum_j p_j a_{ji} + c_i - x_i \right). \tag{6}$$

Starting with $p^0 = \bar{p}$ let

$$p^1 = \Phi(p^0), \ p^2 = \Phi(p^1), ... \tag{7}$$

This iteration yields a monotone decreasing sequence $p^0 \geq p^1 \geq p^2 ...$ . Since it is bounded below it has a limit $p'$, and since $\Phi$ is continuous $p'$ satisfies (3). Hence it is a clearing vector.

We claim that $p'$ is in fact the only solution to (3). Suppose by way of contradiction that there is another clearing vector, say $p'' \neq p'$. As shown by Eisenberg and Noe, the equity values of all nodes must be the same under the two vectors, that is,

$$p'A + (c - x) - p' = p''A + (c - x) - p''.$$

Rearranging terms it follows that

$$(p'' - p')A = p'' - p', \text{ where } p'' - p' \neq 0.$$

This means that the matrix $A$ has eigenvalue 1, which is impossible because under our assumptions $A$ has spectral radius less than 1.

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# 3 Estimating the probability of contagion

Systemic risk can be usefully decomposed into two components: i) the probability that a given set of nodes $D$ will default and ii) the loss in value conditional on $D$ being the default set. This decomposition allows us to distinguish between two distinct phenomena: contagion and amplification. Contagion occurs when defaults by some nodes trigger defaults by other nodes through a domino effect. Amplification occurs when contagion stops but the losses among defaulting nodes keep escalating because of their indebtedness to one another. Roughly speaking the first effect corresponds to a "widening" of the crisis whereas the second corresponds to a "deepening" of the crisis. In this section we shall examine the probability of contagion; the next section deals with the amplification of losses due to network effects.

To estimate the probability of contagion we shall obviously need to make assumptions about the distribution of shocks. We claim, however, that we can estimate the *relative* probability of contagion versus simultaneous default with virtually no information about the network structure and relatively weak conditions on the shock distribution.

To formulate our results we shall need the following notation. Let $\beta_i = (\bar{p}_i - b_i)/\bar{p}_i$ be the proportion of $i$'s liabilities to other entities in the financial system, which we shall call $i$'s *financial connectivity*.[8] We can assume that $\beta_i > 0$ , since otherwise node $i$ would effectively be outside the financial system. Recall that $w_i$ is $i$'s initial net worth (before a shock hits), and $c_i$ is the initial value of its outside assets. The shock $X_i$ to node $i$ takes values in $[0, c_i]$. We shall assume that $w_i > 0$, since otherwise $i$ would already be insolvent. We shall also assume that $w_i \leq c_i$, since otherwise $i$ could never default directly through losses in its own outside assets. Define the ratio $\lambda_i = c_i/w_i \geq 1$ to be the *leverage* of $i$'s outside assets. (This is not the same as $i$'s overall leverage, which in our terminology is the ratio of $i$'s total assets to $i$'s net worth.)

## 3.1 A general bound on the probability of contagion

**Proposition 1.** *Suppose that only node $i$ receives a shock, so that $X_j = 0$ for all $j \neq i$. Suppose that no nodes are in default before the shock. Fix a set of nodes $D$ not containing $i$. The probability that the shock causes all nodes in $D$ to default is at most*

$$P(X_i \geq w_i + (1/\beta_i) \sum_{j \in D} w_j). \tag{8}$$

*Moreover, contagion from $i$ to $D$ is impossible if*

$$\sum_{j \in D} w_j > w_i \beta_i (\lambda_i - 1). \tag{9}$$

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[8]Shin (2012) discusses the reliance of banks on wholesale funding as a contributor to financial crises, and $\beta_i$ measures the degree of this reliance in our setting. Elliott, Golub, and Jackson (2014) propose a related measure which they call the level of integration.

We call the right-hand side of (9) the *contagion index* for $i$. It represents the dollar amount of $i$'s borrowing from other financial institutions, and thus the potential impact on the rest of the financial system if $i$ fails. Inequality (9) says that contagion from $i$ cannot cause all the nodes in a target set $D$ to default if $i$'s contagion index is sufficiently low compared to the aggregate net worth of the nodes in $D$; moreover, this holds regardless of the probability distribution generating the initial shock at $i$. As we shall see in Section 3.2, the contagion index also provides a measure of the *relative likelihood* that the nodes in $D$ default due to direct shocks to their outside assets compared to the likelihood that they default due to contagion from $i$.

Before proceeding to the proof, we illustrate the impossibility condition in (9) through the network in Figure 2(a). The central node is node $i$, meaning that the shock affects its outside assets, and the remaining nodes comprise $D$. The relevant parameter values are $\beta_i = 2/7$, $\lambda_i = 15$, and the net worths are as indicated in the figure. The left side of (9) evaluates to 2 and the right side to 4, so the condition is violated, and, indeed, we saw earlier that contagion is possible with a shock greater than 80. However, a modification of the network that raises the sum of the net worths of the peripheral nodes above 40 makes contagion impossible. This holds, for example, if the outside liabilities of every peripheral node are reduced by more than 5, or if the outside liabilities of a single peripheral node are reduced by more than 20. This example also illustrates that (9) is tight in the sense that if the reverse strict inequality holds, then contagion is possible in this example with a sufficiently large shock.

*Proof of Proposition 1.* Let $D(x) \equiv \bar{D}$ be the default set resulting from the shock vector $X$, whose coordinates are all zero except for $X_i$. By assumption $i$ causes other nodes to default, hence $i$ itself must default, that is, $i \in \bar{D}$. To prove (8) it suffices to show that

$$\beta_i(X_i - w_i) \geq \sum_{j \in \bar{D}-\{i\}} w_j \geq \sum_{j \in D} w_j. \tag{10}$$

The second inequality in (10) follows from the assumption that no nodes are in default before the shock and the fact that we must have $D \subseteq \bar{D} - \{i\}$ for all nodes in $D$ to default.

For the first inequality in (10), define the *shortfall* at node $j$ to be the difference $s_j = \bar{p}_j - p_j$. From (3) we see that the vector of shortfalls $s$ satisfies

$$s = (sA - w + X)_+.$$

By (4) we have $s_j > 0$ for $j \in \bar{D}$ and $s_j = 0$ otherwise. We use a subscript $\bar{D}$ as in $s_{\bar{D}}$ or $A_{\bar{D}}$ to restrict a vector or matrix to the entries corresponding to nodes in the set $\bar{D}$. Then the vector of shortfalls at the nodes of $\bar{D}$ satisfies

$$s_{\bar{D}} \leq s_{\bar{D}}A_{\bar{D}} - w_{\bar{D}} + X_{\bar{D}}, \tag{11}$$

hence

$$X_{\bar{D}} - w_{\bar{D}} \geq s_{\bar{D}}(I_{\bar{D}} - A_{\bar{D}}). \tag{12}$$

The vector $s_{\bar{D}}$ is strictly positive in every coordinate. From the definition of $\beta_j$ we also know that the $j^{th}$ row sum of $I_{\bar{D}} - A_{\bar{D}}$ is at least $1 - \beta_j$. Hence,

$$s_{\bar{D}}(I_{\bar{D}} - A_{\bar{D}}) \cdot 1_{\bar{D}} \geq \sum_{j \in \bar{D}} s_j(1 - \beta_j) \geq s_i(1 - \beta_i). \tag{13}$$

From (11) it follows that the shortfall at node $i$ is at least as large as the initial amount by which $i$ defaults, that is,

$$s_i \geq X_i - w_i > 0. \tag{14}$$

From (12)–(14) we conclude that

$$\sum_{j \in \bar{D}}(X_j - w_j) = X_i - w_i - \sum_{j \in \bar{D}-\{i\}} w_j \geq s_i(1 - \beta_i) \geq (X_i - w_i)(1 - \beta_i). \tag{15}$$

This establishes (10) and the first statement of the proposition. The second statement follows from the first by recalling that the shock to the outside assets cannot exceed their value, that is, $X_i \leq c_i$. Therefore by (8) the probability of contagion is zero if $c_i < w_i + (1/\beta_i)\sum_{j \in D} w_j$. Dividing through by $w_i$ we see that this is equivalent to the condition $\sum_{j \in D} w_j/w_i > \beta_i(\lambda_i - 1)$. $\square$

The preceding proposition relates the probability of contagion from a given node $i$ to the net worth of the defaulting nodes in $D$ relative to $i$'s net worth. The bounds are completely general and do not depend on the distribution of shocks or on the topology of the network. The critical parameters are the financial connectivity, $\beta_i$, and $\lambda_i$, the degree of leverage of $i$'s outside assets.

The Appendix gives estimates of these parameters for large European banks, based on data from stress tests conducted by the European Banking Authority (2011). Among the 50 largest of these banks the average of the $\lambda_i$ is 24.9, the average of our estimated $\beta_i$ is 14.9%, and the average of the products $\beta_i(\lambda_i - 1)$ is 3.2. Proposition 1 implies that contagion from a "typical" bank $i$ cannot topple a set of banks $D$ if the net worth of the latter is more than 3.2 times the net worth of the former, unless there are additional channels of contagion.[9]

## 3.2 Contagion with proportional shocks

We can say a good deal more if we impose some structure on the distribution of shocks. We call contagion from $i$ to $D$ *weak* if the nodes in $D$ are more likely to default through independent direct shocks than

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[9]On average, commercial banks in the United States are leveraged only about half as much as European banks, and their values of $\beta_i$ are somewhat smaller (Federal Reserve Release H.8, Assets and Liabilities of Commercial Banks in the United States, 2012). This suggests that contagion is even less likely in the US financial sector than in Europe.

through contagion from $i$. By Proposition 1, contagion is therefore weak if

$$P(X_i \geq w_i + (1/\beta_i) \sum_{j \in D} w_j) \leq P(X_i > w_i) \prod_{j \in D} P(X_j > w_j). \tag{16}$$

The expression on the left bounds the probability that these nodes default solely through contagion from $i$, while the expression on the right is the probability that the nodes in $D$ default through independent direct shocks.

The assumption of independent direct shocks is unrealistic, but it is conservative. In practice one would expect the shocks to different nodes to be positively associated, so that

$$P(X_i > w_i) \prod_{j \in D} P(X_j > w_j) \leq P(X_i > w_i \text{ and } X_j > w_j, \text{ for all } j \in D).$$

Thus, if contagion is weak when shocks are independent, it is *a fortiori* weak when shocks are positively associated.

Let us assume that the losses at a given node $i$ scale with the size of the portfolio $c_i$. Let us also assume that the distribution of these *relative* losses is the same for all nodes, and independent among nodes. Then there exists a distribution function $H : [0, 1] \rightarrow [0, 1]$ such that

$$F(x_1, ..., x_n) = \prod_{1 \leq i \leq n} H(x_i/c_i). \tag{17}$$

Beta distributions provide a flexible family with which to model the distribution of shocks as a fraction of outside assets. We work with beta densities of form

$$h_{p,q}(y) = \frac{y^{p-1}(1-y)^{q-1}}{B(p,q)}, \quad 0 \leq y \leq 1, \ p, q \geq 1, \tag{18}$$

where $B(p, q)$ is a normalizing constant. The subset with $p = 1$ and $q > 1$ has a decreasing density and seems the most realistic, but (18) is general enough to allow a mode anywhere in the unit interval. The case $q = 1$, $p > 1$ has an increasing density and could be considered "heavy-tailed" in the sense that it assigns greater probability to greater losses, with losses capped at 100 percent of outside assets.[10]

Given a nonempty set of nodes $D$, let $\bar{w}_D$ denote their average net worth. Let $\tilde{\lambda}_D$ denote the harmonic mean of their leverage ratios, that is

$$\tilde{\lambda}_D = \left( \frac{1}{|D|} \sum_{j \in D} \lambda_j^{-1} \right)^{-1}, \tag{19}$$

where $|D|$ denotes the cardinality of the set $D$.

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[10]Bank capital requirements under Basel II and III standards rely on a family of loss distributions derived from a Gaussian copula model. As noted by Tasche (2008) and others, these distributions can be closely approximated by beta distributions.

**Theorem 1.** *Assume the shocks are i.i.d. beta distributed as in* (18) *and that the net worth of every node is initially nonnegative. Let $D$ be a nonempty subset of nodes and let $i \notin D$. Contagion from $i$ to $D$ is weak if*

$$\tilde{\lambda}_D \bar{w}_D \geq w_i \beta_i (\lambda_i - 1). \tag{20}$$

The right-hand side of (20) is $i$'s contagion index, while the left-hand side is a measure of the average vulnerability of the nodes in $D$. Recall that the harmonic mean of any set of positive numbers is less than or equal to their arithmetic mean, hence

$$\tilde{\lambda}_D \leq \bar{\lambda}_D = \frac{1}{|D|} \sum_{j \in D} \lambda_j. \tag{21}$$

It follows from (20) and (21) that contagion from $i$ to $D$ is weak if $\bar{\lambda}_D \bar{w}_D$ is bounded below by $i$'s contagion index. With other parameters held fixed, contagion from $i$ becomes less likely compared to direct default when the nodes in $D$ have larger average net worth or higher average leverage. (The reason for the latter is that higher leverage increases the probability of direct default as well as the probability of indirect default via contagion.) Similarly, contagion from $i$ becomes more likely compared to direct default when $i$ has higher leverage, larger net worth, or higher financial connectivity. In this sense, the contagion index captures the potential risk that specific institutions pose to the rest of the financial system.

A key implication of Theorem 1 is that, without some heterogeneity, contagion is weak *irrespective of the structure* of the interbank network:

**Corollary 1.** *Assume that all nodes hold the same amount of outside assets $c_k \equiv c$. Under the assumptions of Theorem 1, contagion is weak from any node to any other set of nodes, regardless of the network topology.*

*Proof.* Recalling that $\lambda_j^{-1} = w_j/c_j \equiv w_j/c$, we can write

$$\tilde{\lambda}_D = \left[ \left( \frac{1}{|D|} \sum_{j \in D} \frac{w_j}{c} \right) \right]^{-1}. \tag{22}$$

Hence (20) is equivalent to the inequality

$$c \geq w_i \beta_i (\lambda_i - 1). \tag{23}$$

Since $\lambda_i - 1 = c_i/w_i - 1 = c/w_i - 1$, (23) is equivalent to $c \geq \beta_i(c - w_i)$, which always holds because $w_i \geq 0$ and $\beta_i \leq 1$. $\square$

In the Appendix we apply our framework to the 50 largest banks in the stress test data from the European Banking Authority. It turns out that contagion is weak in a wide variety of scenarios. In

particular, we analyze the scenario in which one of the five largest European banks (as measured by assets) topples two other banks in the top 50. We find that the probability of such an event is less than the probability of direct default unless the two toppled banks are near the bottom of the list of 50.

In the example of Figure 2(a), with node $i$ the central node, the left side of (20) evaluates to 20, and the right side evaluates to 16 because $\beta_i = 2/7$, $\lambda_i = 15$, and each of the peripheral nodes has $\lambda_j = 10$. Thus, contagion is weak.

*Proof of Theorem 1.* Proposition 1 implies that contagion is weak from $i$ to $D$ if

$$P(X_i \geq w_i + (1/\beta_i) \sum_{j \in D} w_j) \leq P(X_i > w_i) \prod_{j \in D} P(X_j > w_j). \tag{24}$$

On the one hand this certainly holds if $w_i + (1/\beta_i) \sum_{j \in D} w_j > c_i$, for then contagion is impossible. In this case we obtain, as in (9),

$$\sum_{j \in D} w_j/w_i > \beta_i(\lambda_i - 1). \tag{25}$$

Suppose on the other hand that $(w_i + (1/\beta_i) \sum_{j \in D} w_j) \leq c_i$. By assumption the relative shocks $X_k/c_k$ are independent and beta distributed as in (18). In the uniform case $p = q = 1$, (24) is equivalent to

$$[1 - (w_i/c_i + (1/\beta_i c_i) \sum_{j \in D} w_j)] \leq (1 - w_i/c_i) \prod_{j \in D} (1 - w_j/c_j), \tag{26}$$

We claim that (26) implies (24) for the full family of beta distributions in (18). To see why, first observe that the cumulative distribution $H_{p,q}$ of $h_{p,q}$ satisfies

$$1 - H_{p,q}(y) = H_{q,p}(1 - y).$$

Hence (24) holds if

$$H_{q,p}(1 - w_i/c_i - (1/\beta_i c_i) \sum_{j \in D} w_j) \leq H_{q,p}(1 - w_i/c_i) \prod_{j \in D} H_{q,p}(1 - w_j/c_j). \tag{27}$$

But (27) follows from (26) because beta distributions with $p, q \geq 1$ have the submultiplicative property

$$H_{q,p}(xy) \leq H_{q,p}(x) H_{q,p}(y), \quad x, y \in [0, 1].$$

(See Proposition 4.1.2 of Wirch 1999; the application there has $q \leq 1$, but the proof remains valid for $q \geq 1$. The inequality can also be derived from Corollary 1 of Ramos Romero and Sordo Diaz 2001.) It therefore suffices to establish (27), which is equivalent to

$$(1/\beta_i c_i) \sum_{j \in D} w_j \geq (1 - w_i/c_i)(1 - \prod_{j \in D} (1 - w_j/c_j)). \tag{28}$$

Given any real numbers $\theta_j \in [0, 1]$ we have the inequality

$$\prod_j (1 - \theta_j) \geq 1 - \sum_j \theta_j. \tag{29}$$

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Hence a sufficient condition for (28) to hold is that

$$(1/\beta_i c_i) \sum_{j \in D} w_j \geq (1 - w_i/c_i)(\sum_{j \in D} w_j/c_j), \tag{30}$$

which is equivalent to

$$\begin{aligned}
\sum_{j \in D} w_j &\geq \beta_i(c_i - w_i) \sum_{j \in D} w_j/c_j \\
&= \beta_i w_i(\lambda_i - 1) \sum_{j \in D} \lambda_j^{-1}.
\end{aligned}$$

Expression (20) follows directly from this and formula (19) for the harmonic mean. $\square$

From the argument following (27), it is evident that the same result holds if the shocks to each node $j$ are distributed with parameters $p_j, q_j$ in (18) with $p_i \leq \min_{j \in D} p_j$ and $q_i \geq \max_{j \in D} q_j$.

Of course these results do not say that the network structure has *no* effect on the probability of contagion; indeed there is a considerable literature showing that it does (see among others Haldane and May 2011, Gai and Kapadia 2011, Georg, 2013). Rather it shows that in quite a few situations the probability of contagion will be lower than the probability of direct default, absent some channel of contagion beyond spillovers through payment obligations. We have already mentioned bankruptcy costs, fire sales, and mark-to-market losses as amplifying mechanisms. The models of Demange (2012) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) generate greater contagion by making debts to financial institutions subordinate to other payment obligations. With priority given to outside payments, shocks produce greater losses within the network. In practice, bank debt and bank deposits are owned by both financial institutions and non-financial firms and individuals, so characterizing seniority based on the type of lender is problematic.

## 3.3 Contagion with truncated shocks

In this section we shall show that the preceding results are not an artifact of the beta distribution: similar bounds hold for a variety of shock distributions. Under the beta distribution the probability is zero that a node loses all of its outside assets. One could easily imagine, however, that the probability of this event is positive. This situation can be modeled as follows. Let $X_i^o \geq 0$ be a *primary shock* (potentially unbounded in size) and let $X_i = c_i \wedge X_i^o$ be the resulting loss to $i$'s outside assets. For example $X_i^o$ might represent a loss of income from an employment shock that completely wipes out $i$'s outside assets. Assume that the primary shocks have a joint distribution function of form

$$F^o(x_1^o, ..., x_n^o) = \prod_{1 \leq i \leq n} H^o(x_i^o/c_i), \tag{31}$$

where $H^o$ is a distribution function on the nonnegative real line. In other words we assume that the shocks are i.i.d. and that a given shock $x_i^o$ affects every dollar of outside assets $c_i$ equally. (We shall consider a case of dependent shocks after the next result.)

In general a random variable with distribution function $G$ and density $g$ is said to have an *increasing failure rate (IFR) distribution* if $g(x)/(1 - G(x))$ is an increasing function of $x$. Examples of IFR distributions include all normal, exponential, and uniform distributions and, more generally, all log-concave distributions. Observe that truncating the shock can put mass at $c_i$ and thus assign positive probability to a total loss of outside assets.

**Theorem 2.** *Assume the primary shocks are i.i.d. and IFR-distributed, and that the net worth of every node is initially nonnegative. Let $D$ be a nonempty subset of nodes and let $i \notin D$. Contagion from $i$ to $D$ is weak if*

$$\tilde{\lambda}_D \bar{w}_D > w_i \beta_i \lambda_i. \tag{32}$$

Note that the right-hand side of (32) is a slight variant of the contagion index $w_i \beta_i (\lambda_i - 1)$. From Theorem 1 we know that whenever $\tilde{\lambda}_D \bar{w}_D \geq w_i \beta_i (\lambda_i - 1)$ and the shocks are beta distributed, contagion is weak. Theorem 2 shows that a slightly more stringent inequality implies weak contagion under *any* IFR distribution.

**Corollary 3.** *Assume that all nodes hold the same amount of outside assets $c_i \equiv c$. Under the assumptions of Theorem 2, contagion is weak from any node to any other set of nodes.*

This is immediate upon rewriting (32) as

$$\sum_{j \in D} c_j \lambda_j^{-1} / \sum_{j \in D} \lambda_j^{-1} \geq \beta_i c_i.$$

*Proof of Theorem 2.* Through relabeling, we can assume that the source node for contagion is $i = 1$ and that the infected nodes are $D = \{2, 3, \ldots, m\}$. By Proposition 1 we know that contagion is weak from 1 to $D$ if

$$P(X_1 > w_1 + (1/\beta_1) \sum_{2 \leq j \leq m} w_j) \leq P(X_1 > w_1)P(X_2 > w_2) \cdots P(X_m > w_m). \tag{33}$$

Since $X_1 = c_1 \wedge X_1^o$, the left-hand side is zero when $w_1 + (1/\beta_1) \sum_{2 \leq j \leq m} w_j > c_1$. Thus contagion is impossible if

$$\sum_{2 \leq j \leq m} w_j / w_1 > \beta_1 (\lambda_1 - 1). \tag{34}$$

Let us therefore assume that $w_1 + (1/\beta_1)\sum_{2\leq j\leq m} w_j \leq c_1$. Define the random variables $Y_i = X_i^o/c_i$. Weak contagion from 1 to $D$ holds if

$$P(Y_1 > w_1/c_1 + (1/\beta_1 c_1)\sum_{2\leq j\leq m} w_j) \leq P(Y_1 > w_1/c_1)P(Y_2 > w_2/c_2)\cdots P(Y_m > w_m/c_m)$$

$$= P(Y_1 > w_1/c_1)P(Y_1 > w_2/c_2)\cdots P(Y_1 > w_m/c_m). \tag{35}$$

where the latter follows from the assumption that the $Y_i$ are i.i.d. By assumption $Y_1$ is IFR, hence $P(Y_1 > s + t|Y_1 > s) \leq P(Y_1 > t)$ for all $s, t \geq 0$. (See for example Barlow and Proschan 1975, p.159.) It follows that

$$
\begin{aligned}
&P(Y_1 > \sum_{1\leq k\leq m} w_k/c_k)\\
&= P(Y_1 > w_1/c_1)P(Y_1 > w_1/c_1 + w_2/c_2|Y_1 > w_1/c_1)\\
&\quad \cdots P(Y_1 > w_1/c_1 + w_2/c_2 + \cdots + w_m/c_m|Y_1 > w_1/c_1 + w_2/c_2 + \cdots + w_{m-1}/c_{m-1})\\
&\leq P(Y_1 > w_1/c_1)P(Y_1 > w_2/c_2)\cdots P(Y_1 > w_m/c_m)
\end{aligned}
$$

Together with (35) this shows that contagion from 1 to $D$ is weak provided that

$$P(Y_1 \geq w_1/c_1 + (1/\beta_1 c_1)\sum_{2\leq j\leq m} w_j) \leq P(Y_1 > \sum_{1\leq k\leq m} w_k/c_k). \tag{36}$$

This clearly holds if

$$w_1/c_1 + (1/\beta_1 c_1)\sum_{2\leq j\leq m} w_j \geq \sum_{1\leq k\leq m} w_k/c_k, \tag{37}$$

which is equivalent to

$$(1/\beta_1 c_1)\sum_{2\leq j\leq m} w_j \geq \sum_{2\leq j\leq m} w_j/c_j = \sum_{2\leq j\leq m} \lambda_j^{-1}. \tag{38}$$

Since $c_1 = \lambda_1 w_1$, we can re-write (38) as

$$\sum_{2\leq j\leq m} w_j/w_1 \geq \beta_1\lambda_1 \sum_{2\leq j\leq m} \lambda_j^{-1}. \tag{39}$$

We have therefore shown that if contagion from 1 to $D = \{2, 3, \ldots, m\}$ is possible at all, then (39) is a sufficient condition for weak contagion. $\square$

From (39), we see that a simple sufficient condition for weak contagion is $c_j \geq \beta_1 c_1$, $j = 2, \ldots, m$, and the condition $\sum_{j=2}^{m} w_j > \beta_1(c_1 - w_1)$ makes contagion impossible.

The assumption of independent shocks to different nodes is conservative. If we assume that direct shocks to different nodes are positively dependent, as one would expect in practice, the bounds in Theorems 1 and 2 will be lower and the relative likelihood of default through contagion even smaller.

A particularly simple case arises when the primary shocks $X_i$ are independent with a negative exponential distribution of form

$$f(x_i) = \mu e^{-\mu x_i}, \ x_i \geq 0. \tag{40}$$

If we further assume that $c_1 = \cdots = c_m \equiv c$ and $\beta_1 = 1$, then the two probabilities compared in (35) are equal, both evaluating to $\exp(-\mu \sum_{j \in D} w_j / c)$. In this sense, the exponential distribution is a borderline case in which the probability of a set of defaults from a single shock is roughly equal to the probability from multiple independent shocks. We say "roughly" because the left side of (35) is an upper bound on the probability of default through contagion, and in practice the $\beta_i$ are substantially smaller than 1. In the example of Figure 2(a), we have seen that contagion from the central node requires a shock greater than 80, which has probability $\exp(-80\mu)$ under an exponential distribution. For direct defaults, it suffices to have shocks greater than 5 at the peripheral nodes and a shock greater than 10 at the central node, which has probability $\exp(-30\mu)$ given i.i.d. exponential shocks.

If the primary shocks have a Pareto-like tail, meaning that

$$P(X_i > x) \sim ax^{-\mu} \tag{41}$$

for some positive constants $a$ and $\mu$ (or, more generally, a regularly varying tail), then the probability that a single shock will exceed $\sum_{j \in D} w_j / c$ will be greater than the probability that the nodes in $D$ default through multiple independent shocks, at least at large levels of the $w_j$. However, introducing some dependence can offset this effect, as we now illustrate. To focus on the issue at hand, we take $c = 1$ and $\beta_i = 1$.

To consider a specific and relatively simple case, let $Y_1, \ldots, Y_m$ be independent random variables, each distributed as $t_\nu$, the Student $t$ distribution with $\nu > 2$ degrees of freedom. Let $\tilde{Y}_1, \ldots, \tilde{Y}_m$ have a standard multivariate Student $t$ distribution with $t_\nu$ marginals.[11] The $\tilde{Y}_j$ are uncorrelated but not independent. To make the shocks positive, set $X_j = Y_j^2$ and $\tilde{X}_j = \tilde{Y}_j^2$. Each $X_j$ and $\tilde{X}_j$ has a Pareto-like tail that decays with a power of $\nu/2$.

**Proposition 2.** *With independent shocks $X_j$,*

$$P(X_i > \sum_{j=1}^{m} w_j) \geq \prod_{j=1}^{m} P(X_j > w_j)$$

*for all sufficiently large $w_j$, $j = 1, \ldots, m$. With dependent shocks*

$$P(\tilde{X}_i > \sum_{j=1}^{m} w_j) \leq P(\tilde{X}_j > w_j, j = 1, \ldots, m)$$

*for all $w_j \geq 0$, $j = 1, \ldots, m$.*

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[11]More explicitly, $(\tilde{Y}_1, ..., \tilde{Y}_m)$ has the distribution of $(Z_1, ..., Z_m)/\sqrt{\chi_\nu^2/\nu}$, where the $Z_i$ are independent standard normal random variables and $\chi_v^2$ has a chi-square distribution with $v$ degrees of freedom and is independent of the $Z_i$.

*Proof of Proposition 2.* The first statement follows from applying (41) to both sides of the inequality. The second statement is an application of Bound II for the $F$ distribution on p.1196 of Marshall and Olkin (1974). □

Thus, even with heavy-tailed shocks, we may find that default of a set of nodes through contagion from a single shock is less likely than default through direct shocks to individual nodes if the shocks are dependent.

# 4 Amplification of losses due to network effects

The preceding analysis dealt with the impact of default by a single node (the source) on another set of nodes (the target). That analysis is "partial" in the sense that it does not consider the joint impact of shocks at one more nodes simultaneously. In this section, we examine the impact of shocks on the entire system, including multiple and simultaneous defaults. To carry out such an analysis, we need to have a measure of the total systemic impact of a shock. There appears to be no commonly accepted measure of systemic impact in the prior literature. Eisenberg and Noe (2001) suggest that it is the number of waves of default that a given shock induces in the network. Other authors have suggested that the systemic impact should be measured by the aggregate loss of bank capital; see for example Cont, Moussa, and Santos (2010). Still others have proposed the total loss in value of only those nodes external to the financial sector, i.e. firms and households.

Here we shall take the systemic impact of a shock to be *the direct loss in equity due to the reduction in payments by the external sector to the financial sector, plus the total shortfall in payments from the financial nodes to other financial nodes and to the external sector.* This measure is easily stated in terms of the model variables. Given a shock realization $x$, the systemic impact is

$$\sum_i (x_i \wedge w_i) + S(x) \text{ where } S(x) = \sum_i (\bar{p}_i - p_i(x)). \tag{42}$$

The first term is the direct loss in equity value from reductions in payments by the external sector. The term $S(x)$ is the indirect loss in value from reductions in payments by financial nodes to one another and to the external sector. An overall measure of the riskiness of the system is the *expected loss*

$$L = \int (|x \wedge w| + S(x)) dF(x). \tag{43}$$

The question we wish to examine is what proportion of these losses can be attributed to connections between institutions as opposed to characteristics of individual banks. To analyze this issue let $x$ be a shock and let $D = D(x)$ be the set of nodes that defaults given $x$. Under our assumptions this set is unique because the clearing vector is unique. To avoid notational clutter we shall suppress $x$ in the ensuing discussion.

As in the proof of Propostion 1, define the *shortfall* in payments at node $i$ to be $s_i = \bar{p}_i - p_i$, where $p$ is the clearing vector. By definition of $D$,

$$s_i > 0 \text{ for all } i \in D$$
$$s_i = 0 \text{ for all } i \notin D \tag{44}$$

Also as in the proof of Proposition 1, let $A_D$ be the $|D| \times |D|$ matrix obtained by restricting the relative liabilities matrix $A$ to $D$, and let $I_D$ be the $|D| \times |D|$ identity matrix. Similarly let $s_D$ be the vector of shortfalls $s_i$ corresponding to the nodes in $D$, let $w_D$ be the corresponding net worth vector defined in (1), and let $x_D$ be the corresponding vector of shocks. The clearing condition (3) implies the following *shortfall equation*, provided no node is entirely wiped out — that is, provided $s_i < \bar{p}_i$, for all $i$:

$$s_D A_D - (w_D - x_D) = s_D. \tag{45}$$

Allowing the possibility that some $s_i = \bar{p}_i$, the left side is an upper bound on the right side. Recall that $A_D$ is substochastic, that is, every row sum is at most unity. Moreover, by assumption, there exists a chain of obligations from any given node $k$ to a node having strictly positive obligations to the external sector. It follows that $\lim_{k \to \infty} A_D^k = 0_D$, hence $I_D - A_D$ is invertible and

$$[I_D - A_D]^{-1} = I_D + A_D + A_D^2 + .... \tag{46}$$

From (45) and (46) we conclude that

$$s_D = (x_D - w_D)[I_D + A_D + A_D^2 + ...]. \tag{47}$$

Given a shock $x$ with resulting default set $D = D(x)$, define the vector $u(x) \in R_+^n$ such that

$$u_D(x) = [I_D + A_D + A_D^2 + ...] \cdot 1_D \ , \ u_i(x) = 0 \text{ for all } i \notin D. \tag{48}$$

Combining (42), (47), and (48) shows that total losses given a shock $x$ can be written in the form

$$L(x) = \sum_i (x_i \wedge w_i) + \sum_i (x_i - w_i)u_i(x). \tag{49}$$

The first term represents the direct losses to equity at each node and the second term represents the total shortfall in payments summed over all of the nodes. The right side becomes an upper bound on $L(x)$ if $s_i = \bar{p}_i$ for some $i \in D(x)$.

We call the coefficient $u_i = u_i(x)$ the *depth* of node $i$ in $D = D(x)$. The rationale for this terminology is as follows. Consider a Markov chain on $D$ with transition matrix $A_D$. For each $i \in D$, $u_i$ is the expected number of periods before exiting $D$, starting from node $i$.[12] Expression (48) shows that the node depths measure the *amplification* of losses due to interconnections among nodes in the default set.

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[12]Liu, Feng, and Staum (2016) show that the node depths can be used to characterize the gradient of the clearing vector $p(x)$ with respect to the asset values.

We remark that the concept of node depth is dual to the notion of eigenvector centrality in the networks literature (see for example Newman 2010). To see the connection let us restart the Markov chain uniformly in $D$ whenever it exits $D$. This modified chain has an ergodic distribution proportional to $1_D \cdot [I_D + A_D + A_D^2 + ...]$, and its ergodic distribution measures the centrality of the nodes in $D$. It follows that node depth with respect to $A_D$ corresponds to centrality with respect to the *transpose* of $A_D$.

Although they are related algebraically, the two concepts are quite different. To see why let us return to the example of Figure 2(b). Suppose that node 1 (the central node) suffers a shock $x_1 > 80$. This causes all nodes to default, that is, the default set is $D' = \{1, 2, 3, 4, 5\}$. Consider any node $j > 1$. In the Markov chain described above the expected waiting time to exit the set $D'$, starting from node $j$, is given by the recursion $u_j = 1 + \beta_j u_j$, which implies

$$u_j = 1/(1 - \beta_j) = 1 + y/55. \tag{50}$$

From node 1 the expected waiting time satisfies the recursion

$$u_1 = 1(100/140) + (1 + u_j)(40/140). \tag{51}$$

Hence

$$u_1 = 9/7 + 2y/385. \tag{52}$$

Comparing (50) and (52) we find that node 1 is deeper than the other nodes ($u_1 > u_j$) for $0 \le y < 22$ and shallower than the other nodes for $y > 22$. In contrast, node 1 has lower eigenvector centrality than the other nodes for all $y \ge 0$ because it cannot be reached directly from any other node.

The magnitude of the node depths in a default set can be bounded as follows. In the social networks literature a set $D$ is said to be $\alpha$-*cohesive* if every node in $D$ has at least $\alpha$ of its obligations to other nodes in $D$, that is, $\sum_{j \in D} a_{ij} \ge \alpha$ for every $i \in D$ (Morris, 2000). The *cohesiveness* of $D$ is the maximum such $\alpha$, which we shall denote by $\alpha_D$. From (48) it follows that

$$\forall i \in D, \quad u_i \ge 1/(1 - \alpha_D). \tag{53}$$

Thus the more cohesive the default set, the greater the depth of the nodes in the default set and the greater the amplification of the associated shock.

Similarly we can bound the node depths from above. Recall that $\beta_i$ is the proportion of $i$'s obligations to other nodes in the financial system. Letting $\beta_D = \max\{\beta_i : i \in D\}$ we obtain the upper bound, assuming $\beta_D < 1$,

$$\forall i \in D, \quad u_i \le 1/(1 - \beta_D) . \tag{54}$$

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The bounds in (53) and (54) depend on the default set $D$, which depends on the shock $x$. A *uniform* upper bound is given by

$$\forall i, \quad u_i \leq 1/(1 - \beta^+) \quad \text{where } \beta^+ = \max \beta_i, \tag{55}$$

assuming $\beta^+ < 1$.

We are now in a position to compare the expected systemic losses in a given network of interconnections, and the expected systemic losses without such interconnections. As before, fix a set of $n$ nodes $N = \{1, 2, ..., n\}$, a vector of outside assets $c = (c_1, c_2, ..., c_n) \in R_+^n$ and a vector of outside liabilities $b = (b_1, b_2, ..., b_n) \in R_+^n$. Assume the net worth $w_i$ of node $i$ is nonnegative before a shock is realized. Interconnections are determined by the $n \times n$ liabilities matrix $\bar{P} = (\bar{p}_{ij})$.

Let us compare this situation with the following: eliminate all connections between nodes, that is, let $\bar{P}^o$ be the $n \times n$ matrix of zeroes. Each node $i$ still has outside assets $c_i$ and outside liabilities $b_i$. To keep their net worths unchanged, we introduce "fictitious" outside assets and liabilities to balance the books. In particular if $c_i - b_i < w_i$ we give $i$ a new class of outside assets in the amount $c_i' = w_i - (c_i - b_i)$. If $c_i - b_i > w_i$ we give $i$ a new class of outside liabilities in the amount $b_i' = w_i - (c_i - b_i)$. We shall assume that these new assets are completely safe (they are not subject to shocks), and that the new liabilities have the same priority as all other liabilities.

Let $F(x_1, ..., x_n)$ be a joint shock distribution that is *homogeneous in assets*, that is, $F(x_1, ..., x_n) = G(x_1/c_1, ..., x_n/c_n)$ where $G$ is a symmetric c.d.f. (Unlike in the preceding results on contagion we do not assume that the shocks are independent.) We say that $F$ is IFR if its marginal distributions are IFR; this is equivalent to saying that the marginals of $G$ are IFR. Given $F$, let $\bar{L}$ be the expected total losses in the original network and let $\bar{L}^o$ be the expected total losses when the connections are removed as described above.

**Theorem 3.** *Let $N(b, c, w, \bar{P})$ be a financial system and let $N^o$ be the analogous system with all the connections removed. Assume that the shock distribution is homogeneous in assets and IFR. Let $\beta^+ = \max_i \beta_i < 1$ and let $\delta_i = P(X_i \geq w_i)$. The ratio of expected losses in the original network to the expected losses in $N^o$ is at most*

$$\frac{\bar{L}}{\bar{L}^o} \leq 1 + \frac{\sum \delta_i c_i}{(1 - \beta^+) \sum c_i}. \tag{56}$$

We shall call $1 - \beta^+$ the rate of *dissipation* of the financial system. When $\beta^+$ is close to 1, an initial shock to one institution's outside assets keeps reverberating through the financial system, causing more and more write-downs in asset values until the shock is eventually absorbed. When $\beta^+$ is close to 0, the shock dissipates more rapidly.

Before proceeding with the proof this result, we illustrate it with a numerical example. Our estimate

of $\beta^+$ using the EBA data in the appendix is 0.43. If the marginal default probabilities satisfy $\delta_i \leq 1\%$, then the ratio in (56) is bounded by $1 + (.01)/(1 - 0.43) = 1.0175$, or a 1.75% increase in expected losses in the network compared with the system of isolated nodes. Thus, the theorem shows that if the node-level default probabilities are kept small and the financial connectivity parameters $\beta_i$ are not too close to 1, then the network amplification of losses will be small. This statement holds even when the shocks are dependent, say due to common exposures, and it holds independently of the details of the network structure.

*Proof of Theorem 3.* By assumption the marginals of $F$ are IFR distributed. A general property of IFR distributions is that "new is better than used in expectation," that is,

$$\forall i \; \forall w_i \geq 0, \;\; E[X_i - w_i | X_i \geq w_i] \leq E[X_i] \tag{57}$$

(Barlow and Proschan 1975, p.159). It follows that

$$\forall i \; \forall w_i \geq 0, \;\; E[(X_i - w_i)^+] \leq P(X_i \geq w_i)E[X_i] = \delta_i E[X_i]. \tag{58}$$

By (49) we know that the total expected losses $\bar{L}$ can be bounded as

$$\bar{L} \leq \sum_i E[X_i \wedge w_i] + E[\sum_i (X_i - w_i)u_i(X)]. \tag{59}$$

From (55) we know that $u_i \leq 1/(1 - \beta^+)$ for all $i$; furthermore we clearly have $X_i - w_i \leq (X_i - w_i)^+$ for all $i$. Therefore

$$\bar{L} \leq \sum_i E[X_i \wedge w_i] + (1 - \beta^+)^{-1} \sum_i E[(X_i - w_i)^+]. \tag{60}$$

From this and (58) it follows that

$$\begin{aligned} \bar{L} &\leq & \sum_i E[X_i \wedge w_i] + (1 - \beta^+)^{-1} \sum \delta_i E[X_i] \\ &\leq & \sum_i E[X_i] + (1 - \beta^+)^{-1} \sum \delta_i E[X_i]. \end{aligned} \tag{61}$$

When the network connections are excised, the expected loss is simply the expected sum of the shocks, that is, $\bar{L}^o = \sum_i E[X_i]$. By the assumption of homogeneity in assets we know that $E[X_i] \propto c_i$ for all $i$. We conclude from this and (61) that

$$\bar{L}/\bar{L}^o \leq 1 + \frac{\sum \delta_i c_i}{(1 - \beta^+) \sum c_i}.$$

This completes the proof of Theorem 3. □

The loss measure we introduced in (43) may be viewed as generous in attributing losses to network connections; a more restrictive measure would exclude write-downs of purely financial obligations. This more restrictive measure can be bounded by truncating node depth at 1, which leads to a smaller upper

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bound in Theorem 3. In adopting (43) as our loss measure, we are taking the view that the transactions between financial institutions serve a legitimate purpose and have value, though the nature of these contracts is outside the scope of the model. A write-down in a payment obligation between financial institutions thus represents a loss of value, and chains of payment obligations through a network can then amplify losses when an initial shock leads to a sequence of payment shortfalls along the chain.

# 5 Bankruptcy costs

In this section and the next we enrich the basic framework by incorporating additional mechanisms through which losses propagate from one node to another. We begin by adding bankruptcy costs. The equilibrium condition (3) implicitly assumes that if node $i$'s assets fall short of its liabilities by 1 unit, then the total claims on node $i$ are simply marked down by 1 unit below the face value of $\bar{p}_i$. In practice, the insolvency of node $i$ is likely to produce deadweight losses that have a knock-on effect on the shortfall at node $i$ and at other nodes.

Cifuentes, Ferrucci, and Shin (2005), Battiston et al. (2012), Cont, Moussa, and Santos (2010), and Rogers and Veraart (2012) all use some form of liquidation cost or recovery rate at default in their analyses. Cifuentes, Ferrucci, and Shin (2005) distinguish between liquid and illiquid assets and introduce an external demand function to determine a recovery rate on illiquid assets. Elliott, Golub, and Jackson (2014) attach a fixed cost to bankruptcy. Elsinger, Lehar, and Summer (2006) present simulation results illustrating the effect of bankruptcy costs but without an explicit model. The mechanism we use appears to be the simplest and the closest to the original Eisenberg-Noe setting, which facilitates the analysis of its impact on contagion.

## 5.1 Shortfalls with bankruptcy costs

In the absence of bankruptcy costs, when a node fails its remaining assets are simply divided among its creditors. To capture costs of bankruptcy that go beyond the immediate reduction in payments, we introduce a multiplier $\gamma \geq 0$ and suppose that upon a node's failure its assets are further reduced by

$$\gamma \left[ \bar{p}_i - \left( c_i + \sum_{j \neq i} p_j a_{ji} - x_i \right) \right], \tag{62}$$

up to a maximum reduction at which the assets are entirely wiped out. This approach is analytically tractable and captures the fact that large shortfalls are considerably more costly than small shortfalls, where the firm nearly escapes bankruptcy. The term in square brackets is the difference between node $i$'s obligations $\bar{p}_i$ and its remaining assets. This difference measures the severity of the failure, and the factor $\gamma$ multiplies the severity to generate the knock-on effect of bankruptcy above and beyond the

immediate cost to node $i$'s creditors. We can think of the expression in (62) as an amount of value destroyed or paid out to a fictitious bankruptcy node upon the failure of node $i$.

The resulting condition for a payment vector replaces (3) with

$$p_i = \bar{p}_i \wedge \left[ (1+\gamma)(c_i + \sum_{j \neq i} p_j a_{ji} - x_i) - \gamma \bar{p}_i \right]_+. \tag{63}$$

Written in terms of shortfalls $s_i = \bar{p}_i - p_i$, this becomes

$$s_i = (1+\gamma)[\sum_{j \neq i} s_j a_{ji} - w_i + x_i]_+ \wedge \bar{p}_i. \tag{64}$$

Here we see explicitly how the bankruptcy cost factor $\gamma$ magnifies the shortfalls. Let $\Phi_\gamma$ denote the mapping from the vector $p$ on the right side of (63) to the vector $p$ on the left side.

**Proposition 3.** For each $\gamma \geq 0$, the mapping $\Phi_\gamma$ has a greatest fixed point $p^{(\gamma)}$, and $p^{(\gamma)}$ is decreasing in $\gamma$. The fixed point is unique if $\gamma < (1/\beta^+) - 1$.

*Proof of Proposition 3.* The mapping $p \mapsto \Phi_\gamma(p)$ is monotone decreasing, continuous, and bounded, so the first assertion follows from the argument in Theorem 1 of Eisenberg and Noe (2001). If we set $v_i = c_i + (pA)_i - x_i$, then

$$\Phi_\gamma(p)_i = \begin{cases} v_i - \gamma(\bar{p}_i - v_i), & v_i < \bar{p}_i; \\ \bar{p}_i & \text{otherwise,} \end{cases}$$

so the mapping is monotone in $\gamma$. The maximal fixed-point is the limit of iterations of $\Phi_\gamma$ starting from $\bar{p}$ by the argument in Section 3 of Eisenberg and Noe (2001). If $\gamma_1 \leq \gamma_2$, then the iterates of $\Phi_{\gamma_1}$ are greater than those of $\Phi_{\gamma_2}$, so their maximal fixed-points are ordered the same way. The condition $\gamma < (1/\beta_+) - 1$ implies that the row sums of the matrix $(1+\gamma)A$ are all less than 1, so uniqueness follows as in the case without bankruptcy costs. $\square$

This result confirms that bankruptcy costs expand the set of defaults (i.e., increase contagion) while otherwise leaving the basic structure of the model unchanged. To illustrate the last condition in the proposition, consider again the estimate $\beta_+ = 0.43$ from the EBA data in the appendix. The condition then allows any $\gamma < 1.32$, which is hardly a constraint at all. A multiplier of 1.32 would mean that each dollar of payment shortfall would create an additional 1.32 dollars in bankruptcy costs, above and beyond the shortfall itself. Indeed, even a $\gamma$ value around 0.5, with the immediate impact of increasing the loss by 50% when a node defaults, would be quite large.

In what follows, we examine the amplifying effect of bankruptcy costs conditional on a default set $D$, and we then compare costs with and without network effects. The factor of $1 + \gamma$ in (64) already points to the amplifying effect of bankruptcy costs, but we can take the analysis further.[13]

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[13]A corresponding comparison is possible using partial recoveries at default, as in Rogers and Veraart (2012). This leads to qualitatively similar results, provided claims on other banks are kept to a realistic fraction of a bank's total assets.

Suppose, for simplicity, that the maximum shortfall of $\bar{p}_i$ in (64) is not binding on any of the nodes in the default set. In other words, the shocks are large enough to generate defaults, but not so large as to entirely wipe out asset value at any node. In this case, we have

$$s_D = (1 + \gamma)[s_D A_D - w_D + x_D].$$

If we further assume that $I_D - (1 + \gamma)A_D$ is invertible, then

$$s_D = (1 + \gamma)(x_D - w_D)[I_D - (1 + \gamma)A_D]^{-1}$$

and the systemic shortfall is

$$S(x) = s_D \cdot u_D(\gamma) = (1 + \gamma)(x_D - w_D) \cdot u_D(\gamma), \tag{65}$$

where the modified node depth vector $u_D(\gamma)$ is given by $[I_D - (1 + \gamma)A_D]^{-1} \cdot 1_D$. If $(1 + \gamma)A_D$ has spectral radius less than 1, then

$$u_D(\gamma) = [I_D + (1 + \gamma)A_D + (1 + \gamma)^2 A_D^2 + \cdots] \cdot 1_D.$$

The representation in (65) reveals two effects from introducing bankruptcy costs. The first is an immediate or local impact of multiplying $x_i - w_i$ by $1 + \gamma$; every element of $s_D$ is positive, so the outer factor of $1 + \gamma$ increases the total shortfall. The second and more important effect is through the increased node depth. In particular, letting $\alpha_D$ denote the cohesiveness of $D$ as before, we can now lower-bound the depth of each node by $1/[1 - (1 + \gamma)\alpha_D]$.[14] This makes explicit how bankruptcy costs deepen the losses at defaulted nodes and increase total losses to the system. By the argument in Theorem 3, we get the following comparison of losses with and without interconnections:

**Corollary 4.** *In the setting of Theorem 3, if we introduce bankruptcy costs satisfying $(1 + \gamma)\beta^+ < 1$, then*

$$\frac{\bar{L}}{\bar{L}^o} \leq 1 + \frac{\sum \delta_i c_i}{[1 - (1 + \gamma)\beta^+]\sum_i c_i}. \tag{66}$$

This formula shows that the dissipation rate $1 - \beta^+$ is effectively reduced to $1 - (1 + \gamma)\beta^+$ when we introduce bankruptcy costs, creating greater loss amplification. However, for intermediate values of $\beta^+$, the impact is fairly modest. For example, if as before we take $\beta^+ = 0.43$ and assume $\delta_i \leq 0.01$, and if we set $\gamma$ at the rather large value of 0.5, the corollary gives an upper bound of 1.042. In other words, even with large bankruptcy costs, the additional expected loss attributable to the network is at most 4.2%. Of course, this conclusion depends crucially on the financial connectivity indices $\beta_i$ not being close to 1; the amplification can be very large otherwise.

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[14]If the spectral radius of $(1 + \gamma)A$ is less than 1, then $(1 + \gamma)\alpha_D < 1$.

## 5.2 Example

We saw previously that in the example of Figure 2(a) we need a shock greater than 80 to the outside assets of the central node in order to have contagion to all other nodes. Under a beta distribution with parameters $p = 1$ and $q \geq 1$, this has probability $(1 - 80/150)^q = (7/15)^q$. If we assume i.i.d. proportional shocks to the outside assets of all nodes, then all nodes default directly with probability

$$[(1 - w_1/c_1) \cdots (1 - w_5/c_5)]^q = [(14/15) \cdot 0.9^4]^q \approx 0.61^q > (7/15)^q.$$

Thus contagion is weak.

Now introduce a bankruptcy cost factor of $\gamma = 0.5$ and consider a shock of 57 or larger to the central node. The shock creates a shortfall of at least 47 at the central node that gets magnified by 50% to a shortfall of at least 70.5 after bankruptcy costs. The central node's total liabilities are 140, so the result is that each peripheral node receives less than half of what it is due from the central node, and this is sufficient to push every peripheral node into default. Thus, with bankruptcy costs, the probability of contagion is at least $(1 - (57/150))^q = 0.62^q$, which is now greater than the probability of direct defaults through independent shocks. A similar comparison holds with truncated exponentially distributed shocks.

This example illustrates how bankruptcy costs increase the probability of contagion. However, it is noteworthy that $\gamma$ needs to be quite large to overcome the effect of weak contagion.

## 6  Confidence, credit quality and mark-to-market losses

In the previous section, we demonstrated how bankruptcy costs can amplify losses. In fact, a borrower's deteriorating credit quality can create mark-to-market losses for a lender well before the point of default.[15] Indeed, by some estimates, these types of losses substantially exceeded losses from outright default during 2007-2009. We introduce a mechanism for adding this feature to a network model and show that it too magnifies contagion. Our objective is not to repeat the type of analysis carried out in previous sections but rather to highlight the qualitative differences in this important channel for loss amplification and to suggest that the absence of this mechanism in more conventional models helps explain the limited network effects observed in previous sections.

The mechanism we introduce is illustrated in Figure 3, which shows how the value of node $i$'s total liabilities changes with the level $z$ of node $i$'s assets. The figure shows the special case of a piecewise linear relationship. More generally, let $r(z)$ be the *reduced value* of liabilities at a node as a function of

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[15]See Harris, Herz, and Nissim (2012) for a broad discussion of accounting practices as sources of systemic risk.
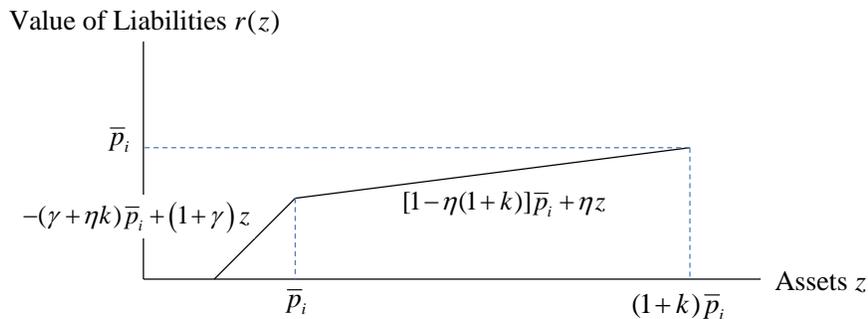
**Figure 3:** Liability value as a function of asset level in the presence of bankruptcy costs and credit quality.

asset level $z$, where $r(z)$ is increasing, continuous, and

$$0 \leq r(z) \leq \bar{p}_i, \quad \text{for } z < (1+k)\bar{p}_i;$$
$$r(z) = \bar{p}_i, \qquad \text{for } z \geq (1+k)\bar{p}_i.$$

Let $R(z_1, \ldots, z_n) = (r(z_1), \ldots, r(z_n))$. Given a shock $x = (x_1, \ldots, x_n)$, the clearing vector $p(z)$ solves

$$p(x) = R(c + p(x)A + w - x).$$

Our conditions on $r$ ensure the existence of such a fixed point by an argument similar to the one in Section 2.2.

The effect of credit quality deterioration begins at a much higher asset level of $z = (1+k)\bar{p}_i$ than does default. Think of $k$ as measuring a capital cushion: node $i$'s credit quality is impaired once its net worth (the difference between its assets and liabilities) falls below the cushion. At this point, the value of $i$'s liabilities begins to decrease, reflecting the mark-to-market impact of $i$'s deteriorating credit quality.

In the example of Figure 2(a), suppose the central node is at its minimum capital cushion before experiencing a shock; in other words $(1+k)\bar{p}_i = 150$, which implies that $k = 1/14$. A shock of 5 to the central node's outside assets reduces the total value of its liabilities by $5\eta$, so each peripheral node incurs a mark-to-market loss of $5\eta/14$. In contrast, in the original model with $\eta = 0$, the peripheral nodes do not experience a loss unless the shock to the central node exceeds 10.

This mechanism reflects an important channel for the spread and amplification of losses that is typically missing in network models. In the baseline model of Section 2, losses cannot spread through the network except through defaults: up until the moment it fails, each node perfectly buffers all other nodes from any shocks it may receive. Bankruptcy costs magnify the consequences of failure but, by definition, operate only once a node has failed. In contrast, declines in credit quality or confidence can propagate losses well before any node has failed. From the perspective of loss transmission, the

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links between nodes in a standard network model are invisible until one of the nodes fails; through the mechanism we have outlined here the links become operative even without defaults, giving the network greater capacity for amplification.

# 7   Concluding Remarks

In this paper, we have shown how simple node-level information can be used to bound contagion and amplification effects in financial networks, without detailed knowledge of the network topology. Our results use three pieces of information about each node: its net worth, its outside leverage, and its financial connectivity. The outside leverage is the ratio of the node's assets outside the network to its net worth; its financial connectivity is the fraction of its liabilities held by other financial institutions. We combine these three quantities into a simple contagion index for each node, and we show that this contagion index determines the relative likelihood that the node will cause other nodes to fail through contagion. Theorem 1 makes this statement precise for a flexible family of shock distributions, and Theorem 2 provides a very similar result for an alternative model of shocks. Importantly, these results hold regardless of the structure of the network. The contagion index thus provides a simple and practical way to monitor an individual institution's potential impact on the rest of the financial system.

Our results imply that it is relatively difficult to generate contagion solely through spillover losses in a network of payment obligations. Network structure matters more for the amplification effect, in which losses among defaulting nodes multiply because of their obligations to one another. In measuring system-wide losses, we include all write-downs in asset values, including all shortfalls in payments to entities both inside and outside the financial network. Within the network, a cycle of payment obligations through a chain of nodes creates a cycle of losses that amplify an initial shock. We characterize the degree of amplification through the concept of node depth, which is the expected number of steps it takes to exit from the default set from a given starting point. We also show that node depth is dual to the concept of eigenvector centrality in the networks literature.

Node depth depends on network structure, but we can bound the expected loss amplification using just node-level information, as we did for contagion. All else equal, a node with higher financial connectivity will spread more of its losses to other nodes within the network when it fails and will have greater node depth. Regardless of the network structure, the node depth for any node is bounded by $1/(1-\beta^{+})$, where $\beta^{+}$ is the maximum financial connectivity over all nodes in the network. Thus, $1-\beta^{+}$ is a measure of how quickly losses dissipate. Theorem 3 uses this quantity to bound loss amplification. More precisely, Theorem 3 bounds the total expected losses in a system with an arbitrary network topology relative to the total expected losses in an otherwise equivalent system of isolated nodes. This result reinforces the importance of our measure of financial connectivity in understanding the potential impact of a financial

network in amplifying losses.

The network structure takes on added importance for both contagion and amplification once we introduce bankruptcy costs and mark-to-market reductions in credit quality. Bankruptcy costs steepen the losses at defaulted nodes, thereby increasing the likelihood that defaults will spread to other nodes. These losses are further amplified by feedback effects, thus increasing the system-wide loss in value; Corollary 4 shows how bankruptcy costs reduce the network's dissipation rate. By contrast, reductions in credit quality have the effect of marking down asset values in advance of default. This process is akin to a slippery slope: once some node suffers a deterioration in its balance sheet, its mark-to-market value decreases, which reduces the value of the nodes to which it has obligations, causing their balance sheets to deteriorate. The result can be a system-wide reduction in value that was triggered solely by a loss of confidence rather than an actual default.

# References

[1] Acemoglu, D., Ozdaglar, A. and Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. *American Economic Review* 105(2), 564–608.

[2] Acharya, V.V., Pedersen, L.H., Philippon, T. and Richardson, M. (2017). Measuring systemic risk. *Review of Financial Studies* 30(1), 2–47.

[3] Adrian, T. and Brunnermeier, M.K. (2016). CoVaR. *American Economic Review* 106(7), 1705–1741.

[4] Allen, F., and Babus, A. (2009). Networks in Finance. In *The Network Challenge: Strategy, Profit, and Risk in an Interlinked World* (eds. P. Kleindorfer, Y. Wind & R. Gunther), Wharton School Publishing, Upper Saddle River, New Jersey.

[5] Allen, F., Babus, A., and Carletti, E. (2012) Asset commonality, debt maturity and systemic risk. *Journal of Financial Economics* 104, pp.519–534.

[6] Allen, F., and Gale, G. (2000). Financial Contagion, *Journal of Political Economy* 108 1–33.

[7] Bank of England (2011). *Financial Stability Report* 29. Available at www.bankofengland.co.uk.

[8] Barlow, R.E., and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing,* Holt, Rinehart, and Winston, New York.

[9] Battiston, S., Delli Gatti, D., Gallegati, M., Greenwald, B., and Stiglitz, J.E. (2012). Default Cascades: When Does Risk Diversification Increase Stability? *Journal of Financial Stability* 8, 138-149.

[10] Caccioli, F., Shrestha, M., Moore, C., and Farmer, J.D. (2012). Stability Analysis of Financial Contagion Due to Overlapping Portfolios. Working Paper 2012-10-018, Santa Fe Institute.

[11] Cifuentes, R., Ferrucci, G., and Shin, H.S. (2005). Liquidity Risk and Contagion, *Journal of the European Economic Association* 3, 556–566.

[12] Cont, R., Moussa, A., and Santos, E. (2010). Network Structure and Systemic Risk in Banking Systems. Available at SSRN: http://ssrn.com/abstract=1733528.

[13] Degryse, H., and Nguyen, G. (2004). Interbank Exposures: An Empirical Examination of Systemic Risk in the Belgian Banking System. Working paper, Belgian National Bank.

[14] Demange, G. (2012). Contagion in Financial Networks: A Threat Index. Working Paper 2012-02, Paris School of Economics, Paris.

[15] Diamond, D.W., and Dybvig, P. (1983). Bank Runs, Deposit Insurance, and Liquidity, *Journal of Political Economy* 91, 401–419.

[16] Eisenberg, L., and Noe, T.H. (2001). Systemic Risk in Financial Systems, *Management Science* 47, 236–249.

[17] Elliott, M., Golub, B. and Jackson, M.O. (2014). Financial networks and contagion. *American Economic Review* 104(10), 3115–3153.

[18] Elsinger, H., Lehar, A., and Summer, M. (2006). Risk Assessment for Banking Systems, *Management Science* 52, 1301–1314.

[19] European Banking Authority (2011), EU-Wide Stress Test Results. Available at www.eba.europa.eu.

[20] Furfine, C. (2003). Interbank Exposures: Quantifying the Risk of Contagion, *Journal of Money, Credit and Banking* 35, 11–128.

[21] Gai, P., Haldane, A., and Kapadia, S. (2011). Complexity, Concentration, and Contagion, *Journal of Monetary Economics* 58, 453–470.

[22] Gai, P., and Kapadia, S. (2010). Contagion in Financial Networks, *Proceedings of the Royal Society A* 466, 2401-2423.

[23] Georg, C.-P. (2013). The Effect of the Interbank Network Structure on Contagion and Common Shocks, *Journal of Banking and Finance* 37, 2216-2228.

[24] Goodhart, C.A.E., Sunirand, P., and Tsomocos, D. P. (2004). A Model to Analyse Financial Fragility: Applications, *Journal of Financial Stability* 1, 1-30.

[25] Haldane, A. G., and May, R. M. (2011). Systemic Risk in Banking Ecosystems, *Nature* 469, 351-355.

[26] Harris, T. S., Herz, R.H., and Nissim, D. (2012) Accounting's Role in the Reporting, Creation, and Avoidance of Systemic Risk in Financial Institutions. In *The Handbook of Systemic Risk* (eds. J.P-Fouqe and J. Langsam), Cambridge University Press.

[27] International Monetary Fund (2012). *Global Financial Stability Report: Restoring Confidence and Progressing on Reforms.* Available at www.imf.org/external/pubs/ft/gfsr.

[28] Staum, J., Feng, M. and Liu, M. (2016). Systemic risk components in a network model of contagion. *IIE Transactions* 48(6), 501–510.

[29] Marshall, A.W., and Olkin, I. (1974). Majorization in Multivariate Distributions. *The Annals of Statistics* 2, 1189-1200.

[30] Morris, S. (2000). Contagion. *Review of Economic Studies* 67, 57-78.

[31] Newman, M. (2010). *Networks: An Introduction.* Oxford University Press, USA.

[32] Nier, E., Yang, J., Yorulmazer, T., and Alentorn, A. (2007) Network Models and Financial Stability, *Journal of Economic Dynamics and Control* 31, 2033-2060.

[33] Office of Financial Research (2012). *Annual Report.* Available at www.treasury.gov/ofr.

[34] Ramos Romer, H.M., and Sordo Diaz, M.A. (2001) The Proportional Likelihood Ratio Order and Applications. *QUESTIIO* 25, 211-223.

[35] Rogers, L.C.G., and Veraart, L.A.M. (2012). Failure and Rescue in an Interbank Network. *Management Science*, in press.

[36] Shin, H. (2012) Global Banking Glut and Loan Risk Premium. *Mundell-Fleming Lecture*, International Monetary Fund, Washington, D.C.

[37] Shleifer, A., and Vishny, R. (2011). Fire Sales in Finance and Economics. *Journal of Economic Perspectives* 25, 29–48.

[38] Summer, M. (2013) Financial contagion and network analysis. *Annual Review of Financial Economics* 5, 11.1–11.21.

[39] Tasche, D. (2008). The Vasicek Distribution. Available at wwwold-m4.ma.tum.de/pers/tasche/.

[40] Upper, C., (2011). Simulation Methods to Assess the Danger of Contagion in Interbank Markets, *Journal of Financial Stability* 7, 111-125.

[41] Upper, C., and Worms, A. (2002). Estimating Bilateral Exposures in the German Interbank Market: Is There a Danger of Contagion? *Discussion Paper 09,* Deutsche Bundesbank.

[42] Vuillemey, G., and Peltonen, T.A. (2013) Sovereign Credit Events and Their Spillovers to the European Banking System — The Interplay Between Sovereign Bonds and CDS Holdings. Working paper, European Central Bank.

[43] Wirch, J.L. (1999) *Coherent Beta Risk Measures for Capital Requirements.* Dissertation, Department of Statistics and Actuarial Sciences, University of Waterloo, Waterloo, Ontario, Canada.

[44] Yellen, J. (2013) Interconnectedness and Systemic Risk: Lessons from the Financial Crisis and Policy Implications. Board of Governors of the Federal Reserve System, Washington, D.C.

# Appendix: Application to European Banks

To provide some insight into how our theoretical results apply in practice, we draw on data from banks that participated in the European Banking Authority's (EBA) 2011 stress test. Detailed information on interbank exposures needed to calibrate a full network model is not publicly available. But our results do not depend on detailed network structure, so the information disclosed with the results of the stress test give us most of what we need.

Ninety banks in 21 countries participated in the stress test. For each, the EBA reports total assets and equity values as of the end of 2010. In addition, the EBA reports each bank's total exposure at default (EAD) to other financial institutions. The EAD measures a bank's total claims on all other banks (not just banks that participated in the stress test and not just European banks), so we take this as the size of each bank's in-network assets. Subtracting this value from total assets gives us $c_i$, the size of the bank's outside assets. For $w_i$ we use the equity values reported by the EBA, which then allows us to calculate $\lambda_i = c_i/w_i$.

The only remaining parameter we need is $\beta_i$, the fraction of a bank's liabilities owed to other banks. This information is not included in the EBA summary, nor is it consistently reported by banks in their financial statements. As a rough indication, we assume that each bank's in-network liabilities equal its in-network assets (though we will see that our results are fairly robust to this assumption).[16] This gives us $\beta_i = $ EAD/(assets-equity).

Some of the smallest banks have problematic data, so as a simple rule we omit the ten smallest. We also omit any countries with only a single participating bank. This leaves us with 76 banks, of which the 50 largest are included in Table 1. For all 76 banks, Table 2 includes assets, EAD, and equity as reported by the EBA (in millions of euros), and our derived values for $c_i, w_i, \beta_i, \lambda_i$. The banks are listed by asset size in Table 1 and grouped by country in Table 2.

In Table 1, we examine the potential for contagion from the failure of each of the five largest banks, BNP Paribas, Deutsche Bank, HSBC, Barclays, and Credit Agricole. Taking each of these in turn as the triggering bank, we then take the default set $D$ to be consecutive pairs of banks. The first default set under BNP Paribas consists of Deutsche Bank and HSBC, the next default set consists of HSBC and Barclays, and so on.

Under "WR" (weak ratio), we report the ratio of the left side of inequality (20) to the right side. Contagion is weak whenever this ratio exceeds 1, as it does in most cases in the table. Contagion fails to be weak only when the banks in the default set are much smaller than the triggering bank. Moreover, the ratio reported for each bank shows how much greater $\beta_i$ would have to be to reverse the direction

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[16]Based on Federal Reserve Release H.8, the average value of $\beta_i$ for commercial banks in the U.S. is about 3%, so our estimates for European banks would appear to be conservative.

of inequality (20). For example, the first ratio listed under BNP Paribas, corresponding to the default of Deutsche Bank and HSBC, is 18.64, based on a $\beta_i$ value of 4.6% (see Table 2). This tells us that the $\beta_i$ value would have to be at least 18.64 x 4.6% = 85.7% for the weak contagion condition to be violated. In this sense, the overall pattern in Table 1 is robust to our estimated values of $\beta_i$. Expanding the default sets generally makes contagion weaker because of the relative magnitudes of $w_i$ and $\lambda_i^{-1}$; see (20) and Table 2.

Under "LR" (for likelihood ratio), we report the relative probability of failure through independent direct shocks and through contagion, calculated as the ratio of the right side of (24) to the left side. This is the ratio of probabilities under the assumption of a uniform distribution $p = q = 1$, which is conservative. An asterisk indicates that the ratio is infinite because default through contagion is impossible.[17] The value of 6.68 reported under BNP Paribas for the default set consisting of Banco Santander and Societe Generale indicates that the probability of default through independent shocks is 6.68 times more likely than default through contagion. Raising the LR values in the table to a power of $q > 1$ gives the corresponding ratio of probabilities under a shock distribution having parameters $p = 1$ and $q$.[18]

Table 2 reports a similar analysis by country. Within each country, we first consider the possibility that the failure of a bank causes the next two largest banks to default – the next two largest banks constitute the default set $D$. (In the case of Belgium, there is only one bank to include in $D$.) For each bank, the column labeled "Ratio for Next Two Banks" reports the ratio of the left side of inequality (20) to the right side. As in Table 1, contagion is weak whenever this ratio exceeds 1. The table shows that the ratio is greater than 1 in every case. As in Table 1, the magnitude of the ratio also tells us how much larger $\beta_i$ would have to be to reverse the inequality in (20).

The last column of the table reports the corresponding test for weak contagion but now holding $D$ fixed as the two smallest banks in each country group. We now see several cases in which the ratio is less than 1 – for example, when we take Deutsche Bank to be the triggering bank and the two smallest banks in the German group (Landesbank Berlin and DekaBank Deutsch Girozentrale) as the default set we get a ratio of 0.5 in the last column.

The results in Tables 1 and 2 reflect the observations we made following Theorem 1 about the relative magnitudes needed for the various model parameters in order that contagion not be weak, and they show that the parameter ranges implying weak contagion are indeed meaningful in practice.

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[17]In deriving (32), we use the bound in (38), which is conservative. Thus, LR must be greater than 1 whenever WR is, and WR can be finite even when LR is infinite and default through contagion is impossible.

[18]We noted previously that a beta distribution can be used to approximate the Gaussian copula loss distribution used in Basel capital standards. We find that the best fit occurs at values of q around 20 or larger. Raising the likelihood ratios in the table to the qth power thus has a major impact.

| | From BNP Paribas | | From Deutsche Bank | | From HSBC | | From Barclays | | From Credit Agricole | |
|---|---|---|---|---|---|---|---|---|---|---|
| | WR | LR | WR | LR | WR | LR | WR | LR | WR | LR |
| BNP PARIBAS | | | | | | | | | | |
| DEUTSCHE BANK AG | | | | | | | | | | |
| HSBC HOLDINGS plc | 18.64 | * | | | | | | | | |
| BARCLAYS plc | 18.63 | * | 9.21 | 3.89 | | | | | | |
| CREDIT AGRICOLE | 17.83 | * | 8.81 | 2.01 | 8.27 | 1.88 | 27.23 | | | |
| BANCO SANTANDER S.A. | 14.91 | * | 7.37 | 1.98 | 6.92 | 1.85 | 24.62 | * | | |
| SOCIETE GENERALE | 12.52 | 6.68 | 6.19 | 1.62 | 5.81 | 1.55 | 20.67 | * | 13.66 | 15.32 |
| LLOYDS BANKING GROUP plc | 11.23 | 7.70 | 5.55 | 1.63 | 5.21 | 1.56 | 18.55 | * | 12.26 | 23.52 |
| BPCE | 11.29 | 12.77 | 5.58 | 1.70 | 5.24 | 1.62 | 18.65 | * | 12.32 | * |
| ING BANK NV | 10.32 | 3.44 | 5.10 | 1.46 | 4.79 | 1.41 | 17.05 | * | 11.27 | 4.57 |
| UNICREDIT S.p.A | 9.55 | 4.06 | 4.72 | 1.49 | 4.43 | 1.44 | 15.77 | * | 10.42 | 5.90 |
| COMMERZBANK AG | 8.47 | 3.33 | 4.18 | 1.43 | 3.93 | 1.38 | 13.98 | * | 9.24 | 4.39 |
| RABOBANK NEDERLAND | 6.96 | 2.48 | 3.44 | 1.33 | 3.23 | 1.29 | 11.49 | * | 7.59 | 2.94 |
| ROYAL BANK OF SCOTLAND GROUP plc | 6.06 | * | 2.99 | 1.67 | 2.81 | 1.58 | 10.01 | * | 6.61 | * |
| INTESA SANPAOLO S.p.A | 5.70 | 72.67 | 2.82 | 1.63 | 2.64 | 1.54 | 9.41 | * | 6.22 | * |
| DEXIA | 4.59 | 1.79 | 2.27 | 1.19 | 2.13 | 1.16 | 7.58 | 5.19 | 5.01 | 1.97 |
| Nordea Bank AB (publ) | 4.52 | 1.57 | 2.23 | 1.15 | 2.10 | 1.13 | 7.46 | 2.96 | 4.93 | 1.68 |
| BANCO BILBAO VIZCAYA ARGENTARIA | 5.24 | 1.85 | 2.59 | 1.21 | 2.43 | 1.19 | 8.65 | 5.82 | 5.72 | 2.05 |
| DANSKE BANK | 4.47 | 1.66 | 2.21 | 1.16 | 2.08 | 1.14 | 7.39 | 3.72 | 4.88 | 1.80 |
| ABN AMRO BANK NV | 3.91 | 1.33 | 1.93 | 1.09 | 1.81 | 1.08 | 6.46 | 1.85 | 4.27 | 1.38 |
| Landesbank Baden-Württemberg | 3.36 | 1.23 | 1.66 | 1.06 | 1.56 | 1.05 | 5.55 | 1.57 | 3.67 | 1.27 |
| Hypo Real Estate Holding AG, München | 3.00 | 1.15 | 1.48 | 1.03 | 1.39 | 1.03 | 4.96 | 1.34 | 3.28 | 1.17 |
| BFA-BANKIA | 3.38 | 1.21 | 1.67 | 1.05 | 1.57 | 1.04 | 5.59 | 1.49 | 3.69 | 1.24 |
| DZ BANK AG Dt. Zentral-Genossenschaft | 2.83 | 1.21 | 1.40 | 1.04 | 1.31 | 1.03 | 4.67 | 1.54 | 3.08 | 1.25 |
| Bayerische Landesbank | 2.39 | 1.16 | 1.18 | 1.02 | 1.11 | 1.01 | 3.94 | 1.42 | 2.61 | 1.20 |
| KBC BANK | 2.73 | 1.24 | 1.35 | 1.04 | 1.27 | 1.03 | 4.50 | 1.63 | 2.98 | 1.28 |
| CAJA DE AHORROS Y PENSIONES DE BA | 3.03 | 1.24 | 1.50 | 1.05 | 1.41 | 1.04 | 5.01 | 1.63 | 3.31 | 1.29 |
| BANCA MONTE DEI PASCHI DI SIENA S | 2.96 | 1.17 | 1.46 | 1.04 | 1.38 | 1.03 | 4.89 | 1.40 | 3.23 | 1.20 |
| Svenska Handelsbanken AB (publ) | 2.61 | 1.13 | 1.29 | 1.02 | 1.21 | 1.02 | 4.31 | 1.30 | 2.85 | 1.15 |
| Norddeutsche Landesbank -GZ | 2.13 | 1.09 | 1.05 | 1.00 | 0.99 | 1.00 | 3.52 | 1.22 | 2.33 | 1.11 |
| Skandinaviska Enskilda Banken AB (publ) | 1.96 | 1.09 | 0.97 | 1.00 | 0.91 | 0.99 | 3.24 | 1.25 | 2.14 | 1.11 |
| DnB NOR Bank ASA | 2.26 | 1.16 | 1.12 | 1.02 | 1.05 | 1.01 | 3.73 | 1.44 | 2.46 | 1.20 |
| Erste Bank Group (EBG) | 2.22 | 1.17 | 1.10 | 1.01 | 1.03 | 1.01 | 3.66 | 1.47 | 2.42 | 1.21 |
| WestLB AG, Düsseldorf | 1.91 | 1.10 | 0.94 | 1.00 | 0.88 | 0.99 | 3.15 | 1.27 | 2.08 | 1.12 |
| Swedbank AB (publ) | 1.82 | 1.07 | 0.90 | 0.99 | 0.84 | 0.99 | 3.00 | 1.19 | 1.98 | 1.09 |
| Nykredit | 1.98 | 1.10 | 0.98 | 1.00 | 0.92 | 0.99 | 3.27 | 1.26 | 2.16 | 1.12 |
| BANK OF IRELAND | 1.76 | 1.08 | 0.87 | 0.99 | 0.82 | 0.98 | 2.91 | 1.24 | 1.92 | 1.10 |
| HSH Nordbank AG, Hamburg | 1.63 | 1.06 | 0.80 | 0.98 | 0.76 | 0.98 | 2.69 | 1.18 | 1.78 | 1.08 |
| BANCO POPOLARE - S.C. | 1.58 | 1.05 | 0.78 | 0.98 | 0.73 | 0.98 | 2.61 | 1.15 | 1.73 | 1.06 |
| Landesbank Berlin AG | 1.21 | 1.03 | 0.60 | 0.96 | 0.56 | 0.95 | 1.99 | 1.13 | 1.32 | 1.04 |
| ALLIED IRISH BANKS PLC | 1.12 | 1.01 | 0.55 | 0.96 | 0.52 | 0.96 | 1.85 | 1.10 | 1.22 | 1.03 |
| Raiffeisen Bank International (RBI) | 1.23 | 1.03 | 0.61 | 0.96 | 0.57 | 0.95 | 2.04 | 1.14 | 1.35 | 1.05 |
| UNIONE DI BANCHE ITALIANE SCPA (U | 1.22 | 1.04 | 0.60 | 0.95 | 0.57 | 0.94 | 2.02 | 1.19 | 1.33 | 1.06 |
| DekaBank Deutsche Girozentrale, Fran | 1.19 | 1.02 | 0.59 | 0.96 | 0.55 | 0.96 | 1.96 | 1.12 | 1.30 | 1.04 |
| BANCO POPULAR ESPAÑOL, S.A. | 1.21 | 1.03 | 0.60 | 0.96 | 0.56 | 0.96 | 2.00 | 1.12 | 1.32 | 1.04 |
| CAIXA GERAL DE DEPÓSITOS, SA | 1.27 | 1.04 | 0.63 | 0.96 | 0.59 | 0.95 | 2.10 | 1.18 | 1.39 | 1.06 |
| NATIONAL BANK OF GREECE | 1.25 | 1.05 | 0.62 | 0.95 | 0.58 | 0.94 | 2.07 | 1.21 | 1.37 | 1.07 |
| BANCO COMERCIAL PORTUGUÊS, SA ( | 1.21 | 1.03 | 0.60 | 0.95 | 0.56 | 0.95 | 2.00 | 1.15 | 1.32 | 1.05 |
| BANCO DE SABADELL, S.A. | 1.08 | 1.01 | 0.53 | 0.96 | 0.50 | 0.96 | 1.78 | 1.07 | 1.17 | 1.02 |
| EFG EUROBANK ERGASIAS S.A. | 1.01 | 1.00 | 0.50 | 0.95 | 0.47 | 0.95 | 1.66 | 1.07 | 1.10 | 1.01 |
| ESPÍRITO SANTO FINANCIAL GROUP, S | 0.92 | 0.99 | 0.46 | 0.94 | 0.43 | 0.94 | 1.52 | 1.07 | 1.01 | 1.00 |

Table 1: Results based on 2011 EBA stress test data. We consider contagion from each of the five largest banks across the top to pairs of banks in consecutive rows. A weak ratio (WR) value greater than 1 indicates that contagion is weak. Each LR value is a likelihood ratio for default through independent shocks to default through contagion, assuming a uniform fractional shock.

| | Bank Number and Name | Assets | EAD | c_i<br>Outside Assets | w_i<br>Equity | beta_i<br>EAD/(Assets-Equity) | lambda_i<br>Outside Assets/Equity | Ratio for Next Two Banks | Ratio for Last Two Banks |
|---|---|---|---|---|---|---|---|---|---|
| AT001 | Erste Bank Group (EBG) | 205,938 | 25,044 | 180,894 | 10,507 | 12.8% | 17.2 | | |
| AT002 | Raiffeisen Bank International (RBI) | 131,173 | 30,361 | 100,812 | 7,641 | 24.6% | 13.2 | | |
| AT003 | Oesterreichische Volksbank AG | 44,745 | 10,788 | 33,957 | 1,765 | 25.1% | 19.2 | 3.4 | |
| BE004 | DEXIA | 548,135 | 228,211 | 319,924 | 17,002 | 43.0% | 18.8 | | |
| BE005 | KBC BANK | 276,723 | 23,871 | 252,852 | 11,705 | 9.0% | 21.6 | 35.7 | |
| CY006 | MARFIN POPULAR BANK PUBLIC CO LTI | 42,580 | 7,907 | 34,673 | 2,015 | 19.5% | 17.2 | | |
| CY007 | BANK OF CYPRUS PUBLIC CO LTD | 41,996 | 7,294 | 34,702 | 2,134 | 18.3% | 16.3 | 1.6 | |
| DE017 | DEUTSCHE BANK AG | 1,905,630 | 194,399 | 1,711,231 | 30,361 | 10.4% | 56.4 | | 0.5 |
| DE018 | COMMERZBANK AG | 771,201 | 138,190 | 633,011 | 26,728 | 18.6% | 23.7 | | 0.8 |
| DE019 | Landesbank Baden-Württemberg | 374,413 | 133,906 | 240,507 | 9,838 | 36.7% | 24.4 | 2.5 | 1.0 |
| DE020 | DZ BANK AG Dt. Zentral-Genossenscha | 323,578 | 135,860 | 187,718 | 7,299 | 43.0% | 25.7 | 1.9 | 1.1 |
| DE021 | Bayerische Landesbank | 316,354 | 97,336 | 219,018 | 11,501 | 31.9% | 19.0 | 2.4 | 1.3 |
| DE022 | Norddeutsche Landesbank -GZ | 228,586 | 91,217 | 137,369 | 3,974 | 40.6% | 34.6 | 2.5 | 1.6 |
| DE023 | Hypo Real Estate Holding AG, München | 328,119 | 29,084 | 299,035 | 5,539 | 9.0% | 54.0 | 3.0 | 3.3 |
| DE024 | WestLB AG, Düsseldorf | 191,523 | 58,128 | 133,395 | 4,218 | 31.0% | 31.6 | 3.6 | 2.2 |
| DE025 | HSH Nordbank AG, Hamburg | 150,930 | 9,532 | 141,398 | 4,434 | 6.5% | 31.9 | 5.2 | 9.7 |
| DE027 | Landesbank Berlin AG | 133,861 | 49,253 | 84,608 | 5,162 | 38.3% | 16.4 | 2.6 | |
| DE028 | DekaBank Deutsche Girozentrale, Fran | 130,304 | 41,255 | 89,049 | 3,359 | 32.5% | 26.5 | 9.7 | |
| DK008 | DANSKE BANK | 402,555 | 75,894 | 326,661 | 14,576 | 19.6% | 22.4 | | 0.4 |
| DK011 | Nykredit | 175,888 | 8,597 | 167,291 | 6,633 | 5.1% | 25.2 | | 2.7 |
| DK009 | Jyske Bank | 32,752 | 4,674 | 28,078 | 1,699 | 15.1% | 16.5 | 1.4 | |
| DK010 | Sydbank | 20,238 | 3,670 | 16,568 | 1,231 | 19.3% | 13.5 | 2.7 | |
| ES059 | BANCO SANTANDER S.A. | 1,223,267 | 51,407 | 1,171,860 | 41,998 | 4.4% | 27.9 | | 0.4 |
| ES060 | BANCO BILBAO VIZCAYA ARGENTARIA | 540,936 | 110,474 | 430,462 | 24,939 | 21.4% | 17.3 | | 0.2 |
| ES061 | BFA-BANKIA | 327,930 | 39,517 | 288,414 | 13,864 | 12.6% | 20.8 | 7.4 | 0.6 |
| ES062 | CAJA DE AHORROS Y PENSIONES DE BA | 275,856 | 5,510 | 270,346 | 11,109 | 2.1% | 24.3 | 3.2 | 3.6 |
| ES064 | BANCO POPULAR ESPAÑOL, S.A. | 129,183 | 14,810 | 114,373 | 6,699 | 12.1% | 17.1 | 5.2 | 1.5 |
| ES065 | BANCO DE SABADELL, S.A. | 96,703 | 3,678 | 93,025 | 3,507 | 3.9% | 26.5 | 19.6 | 5.5 |
| ES066 | CAIXA D'ESTALVIS DE CATALUNYA, TAR | 76,014 | 8,219 | 67,795 | 3,104 | 11.3% | 21.8 | 6.1 | 2.7 |
| ES067 | CAIXA DE AFORROS DE GALICIA, VIGO, | 73,319 | 2,948 | 70,371 | 2,849 | 4.2% | 24.7 | 19.5 | 6.9 |
| ES083 | CAJA DE AHORROS DEL MEDITERRANE | 72,034 | 4,981 | 67,053 | 1,843 | 7.1% | 36.4 | 9.5 | 4.2 |
| ES071 | GRUPO BANCA CIVICA | 71,055 | 7,419 | 63,636 | 3,688 | 11.0% | 17.3 | 22.9 | 3.0 |
| ES068 | GRUPO BMN | 69,760 | 7,660 | 62,101 | 3,304 | 11.5% | 18.8 | 13.6 | 2.9 |
| ES063 | EFFIBANK | 54,523 | 4,124 | 50,399 | 2,656 | 8.0% | 19.0 | 8.5 | 5.2 |
| ES069 | BANKINTER, S.A. | 53,476 | 2,141 | 51,335 | 1,920 | 4.2% | 26.7 | 7.5 | 9.5 |
| ES070 | CAJA ESPAÑA DE INVERSIONES, SALAM | 45,656 | 7,235 | 38,422 | 2,076 | 16.6% | 18.5 | 11.5 | 3.2 |
| ES075 | GRUPO BBK | 44,628 | 1,924 | 42,704 | 2,982 | 4.6% | 14.3 | 19.9 | 10.7 |
| ES072 | CAJA DE AHORROS Y M.P. DE ZARAGOZ | 42,716 | 1,779 | 40,938 | 2,299 | 4.4% | 17.8 | 6.9 | 11.5 |
| ES073 | MONTE DE PIEDAD Y CAJA DE AHORRO | 34,263 | 2,599 | 31,664 | 2,501 | 8.2% | 12.7 | 19.4 | 8.2 |
| ES074 | BANCO PASTOR, S.A. | 31,135 | 1,665 | 29,470 | 1,395 | 5.6% | 21.1 | 18.1 | 12.5 |
| ES076 | CAIXA D'ESTALVIS UNIO DE CAIXES DE | 28,310 | 1,802 | 26,508 | 1,065 | 6.6% | 24.9 | 11.8 | 11.6 |
| ES077 | CAJA DE AHORROS Y M.P. DE GIPUZKO | 20,786 | 285 | 20,501 | 1,935 | 1.5% | 10.6 | 14.2 | |
| ES078 | GRUPO CAJA3 | 20,144 | 1,926 | 18,218 | 1,164 | 10.1% | 15.7 | 11.6 | |
| FR013 | BNP PARIBAS | 1,998,157 | 90,328 | 1,907,829 | 55,352 | 4.6% | 34.5 | | 11.1 |
| FR014 | CREDIT AGRICOLE | 1,503,621 | 83,713 | 1,419,908 | 46,277 | 5.7% | 30.7 | | 12.2 |
| FR016 | SOCIETE GENERALE | 1,051,323 | 100,013 | 951,310 | 27,824 | 9.8% | 34.2 | 13.9 | |
| FR015 | BPCE | 1,000,695 | 34,983 | 965,712 | 31,943 | 3.6% | 30.2 | 12.2 | |
| GB089 | HSBC HOLDINGS plc | 1,783,199 | 212,092 | 1,571,107 | 86,900 | 12.5% | 18.1 | | 3.5 |
| GB090 | BARCLAYS plc | 1,725,709 | 53,873 | 1,671,836 | 46,232 | 3.2% | 36.2 | | 12.3 |
| GB091 | LLOYDS BANKING GROUP plc | 1,006,082 | 29,233 | 976,849 | 47,984 | 3.1% | 20.4 | 6.6 | |
| GB088 | ROYAL BANK OF SCOTLAND GROUP plc | 607,151 | 105,506 | 501,845 | 58,982 | 19.2% | 8.5 | 12.3 | |
| GR031 | NATIONAL BANK OF GREECE | 118,832 | 8,608 | 110,224 | 8,153 | 7.8% | 13.5 | | 6.0 |
| GR030 | EFG EUROBANK ERGASIAS S.A. | 85,885 | 3,838 | 82,047 | 4,296 | 4.7% | 19.1 | | 12.9 |
| GR032 | ALPHA BANK | 66,798 | 3,492 | 63,306 | 5,275 | 5.7% | 12.0 | 8.9 | 14.4 |
| GR033 | PIRAEUS BANK GROUP | 57,680 | 1,581 | 56,099 | 3,039 | 2.9% | 18.5 | 16.5 | |
| GR034 | AGRICULTURAL BANK OF GREECE S.A. ( | 31,221 | 1,657 | 29,564 | 792 | 5.4% | 37.3 | 14.4 | |
| IE038 | BANK OF IRELAND | 156,712 | 17,254 | 139,458 | 7,037 | 11.5% | 19.8 | | 4.9 |
| IE037 | ALLIED IRISH BANKS PLC | 131,311 | 11,277 | 120,034 | 3,669 | 8.8% | 32.7 | | |
| IE039 | IRISH LIFE AND PERMANENT | 46,743 | 6,127 | 40,616 | 1,681 | 13.6% | 24.2 | 4.9 | |
| IT041 | UNICREDIT S.p.A | 929,488 | 106,707 | 822,781 | 35,702 | 11.9% | 23.0 | | 1.3 |
| IT040 | INTESA SANPAOLO S.p.A | 576,962 | 109,909 | 467,053 | 26,159 | 20.0% | 17.9 | | 1.4 |
| IT042 | BANCA MONTE DEI PASCHI DI SIENA S. | 244,279 | 12,074 | 232,205 | 6,301 | 5.1% | 36.9 | 4.2 | 10.4 |
| IT043 | BANCO POPOLARE - S.C. | 140,043 | 7,602 | 132,440 | 5,474 | 5.6% | 24.2 | 2.0 | |
| IT044 | UNIONE DI BANCHE ITALIANE SCPA (UE | 130,559 | 19,793 | 110,766 | 6,559 | 16.0% | 16.9 | 10.4 | |
| NL047 | ING BANK NV | 933,073 | 111,756 | 821,317 | 30,895 | 12.0% | 26.6 | | 2.5 |
| NL048 | RABOBANK NEDERLAND | 607,483 | 37,538 | 569,945 | 27,725 | 6.2% | 20.6 | | 7.2 |
| NL049 | ABN AMRO BANK NV | 379,599 | 29,196 | 350,403 | 11,574 | 7.7% | 30.3 | 5.1 | |
| NL050 | SNS BANK NV | 78,918 | 388 | 78,530 | 1,782 | 0.5% | 44.1 | 7.2 | |
| PT053 | CAIXA GERAL DE DEPÓSITOS, SA | 119,334 | 14,221 | 105,097 | 6,510 | 11.9% | 16.1 | | 5.0 |
| PT054 | BANCO COMERCIAL PORTUGUÊS, SA (E | 100,010 | 7,690 | 92,320 | 3,521 | 7.7% | 26.2 | | 8.5 |
| PT055 | ESPÍRITO SANTO FINANCIAL GROUP, SA | 85,644 | 8,690 | 76,955 | 4,520 | 10.1% | 17.0 | 7.1 | |
| PT056 | Banco BPI, SA | 43,826 | 5,463 | 38,364 | 2,133 | 12.5% | 18.0 | 8.5 | |
| SE084 | Nordea Bank AB (publ) | 542,853 | 61,448 | 481,405 | 19,103 | 11.3% | 25.2 | | 3.7 |
| SE085 | Skandinaviska Enskilda Banken AB (pub | 212,240 | 25,955 | 186,284 | 9,604 | 12.2% | 19.4 | | 9.0 |
| SE086 | Svenska Handelsbanken AB (publ) | 240,202 | 20,870 | 219,332 | 8,209 | 8.7% | 26.7 | 3.8 | |
| SE087 | Swedbank AB (publ) | 191,365 | 17,358 | 174,007 | 7,352 | 9.1% | 23.7 | 9.0 | |

Table 2: Results based on 2011 EBA stress test data. In the last two columns, a ratio greater than 1 indicates weak contagion from a bank to the next two largest banks in the same country and to the two smallest banks in the country group, respectively.